

AN OVERVIEW OF THE QUANTIZATION FOR MIXED DISTRIBUTIONS

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ABSTRACT. The basic goal of quantization for probability distribution is to reduce the number of values, which is typically uncountable, describing a probability distribution to some finite set and thus approximation of a continuous probability distribution by a discrete distribution. Mixed distributions are an exciting new area for optimal quantization. In this paper, we have determined the optimal sets of n -means, the n th quantization error, and the quantization dimensions of different mixed distributions. Besides, we have discussed whether the quantization coefficients for the mixed distributions exist. The results in this paper will give a motivation and insight into more general problems in quantization for mixed distributions.

1. INTRODUCTION

The quantization problem for a probability distribution has a deep background in information theory such as signal processing and data compression (see [GG, GN, Z]). Although the work of quantization in engineering science has a long history, rigorous mathematical treatment has given by Graf and Luschgy (see [GL1]). Let us consider a Borel probability measure P on \mathbb{R}^d and a natural number $n \in \mathbb{N}$. Then, the n th *quantization error* for P is defined by:

$$V_n := V_n(P) = \inf \left\{ \int \min_{a \in \alpha} \|x - a\|^2 dP(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d . A set α for which the infimum occurs and contains no more than n points is called an *optimal set of n -means*, or *optimal set of n -quantizers*. Of course, this makes sense only if the mean squared error or the expected squared Euclidean distance $\int \|x\|^2 dP(x)$ is finite (see [AW, GKL, GL, GL1]). It is known that for a continuous probability measure an optimal set of n -means always has exactly n -elements (see [GL1]). For a finite set $\alpha \subset \mathbb{R}^d$, the number $\int \min_{a \in \alpha} \|x - a\|^2 dP(x)$ is often referred to as the *cost* or *distortion error* for α with respect to the probability distribution P . The numbers

$$\underline{D}(P) := \liminf_{n \rightarrow \infty} \frac{2 \log n}{-\log V_n(P)}, \text{ and } \overline{D}(P) := \limsup_{n \rightarrow \infty} \frac{2 \log n}{-\log V_n(P)},$$

are, respectively, called the *lower* and *upper quantization dimensions* of the probability measure P . If $\underline{D}(P) = \overline{D}(P)$, the common value is called the *quantization dimension* of P and is denoted by $D(P)$. For any $s \in (0, +\infty)$, the numbers $\liminf_n n^{\frac{2}{s}} V_n(P)$ and $\limsup_n n^{\frac{2}{s}} V_n(P)$ are, respectively, called the *s -dimensional lower* and *upper quantization coefficients* for P . If the s -dimensional lower and upper quantization coefficients for P are finite and positive, then s coincides with the quantization dimension of P . Main concerns in quantization problem include (i) the asymptotic properties of the quantization errors such as the quantization dimensions and the quantization coefficients; (ii) the optimal sets in the quantization for a given measure. It is known that for any Borel probability measure P on \mathbb{R}^d with non-vanishing absolutely continuous part $\lim_n n^{\frac{2}{d}} V_n(P)$ is finite and strictly positive (see [BW]); in other words, the quantization dimension of a Borel probability measure with non-vanishing absolutely continuous part equals the dimension d of the underlying space. Although absolutely continuous probability measures have been well studied, there are not many results on the optimal sets for such a measure. In

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fact, to determine the optimal sets for a probability measure, singular or nonsingular, is much more difficult than to determine the quantization dimension of such a measure. For some work in the direction of optimal sets for a probability measure, one can see [DR, GL2, R1, R2]. For a finite set $\alpha \subset \mathbb{R}^d$, the *Voronoi region* generated by $a \in \alpha$, denoted by $M(a|\alpha)$, is defined to be the set of all elements in \mathbb{R}^d which are nearest to a . The set $\{M(a|\alpha) : a \in \alpha\}$ is called the *Voronoi diagram* or *Voronoi tessellation* of \mathbb{R}^d with respect to α . The point a is called the centroid of its own Voronoi region if $a = E(X : X \in M(a|\alpha))$, where X is a P -distributed random variable. Let us now state the following proposition (see [GG, GL1]).

Proposition 1.1. *Let α be an optimal set of n -means, $a \in \alpha$, and $M(a|\alpha)$ be the Voronoi region generated by $a \in \alpha$. Then, for every $a \in \alpha$, (i) $P(M(a|\alpha)) > 0$, (ii) $P(\partial M(a|\alpha)) = 0$, (iii) $a = E(X : X \in M(a|\alpha))$, and (iv) P -almost surely the set $\{M(a|\alpha) : a \in \alpha\}$ forms a Voronoi partition of \mathbb{R}^d .*

(p_1, p_2, \dots, p_N) is a probability vector, by that it is meant that $0 < p_j < 1$ for all $1 \leq j \leq N$, and $\sum_{j=1}^N p_j = 1$. We now give the following definition.

Definition 1.2. Let P_1, P_2, \dots, P_N be Borel probability measures on \mathbb{R}^d , and (p_1, p_2, \dots, p_N) be a probability vector. Then, a Borel probability measure P on \mathbb{R}^d is called a *mixed probability distribution*, or in short, *mixed distribution*, generated by P_1, P_2, \dots, P_N and the probability vector if for all Borel subsets A of \mathbb{R}^d , $P(A) = p_1 P_1(A) + p_2 P_2(A) + \dots + p_N P_N(A)$. Such a mixed distribution is denoted by $P := p_1 P_1 + p_2 P_2 + \dots + p_N P_N$, and P_1, P_2, \dots, P_N are called the *components* of the mixed distribution.

In this paper, in Section 2, we have considered a mixed distribution $P := pP_1 + (1-p)P_2$, where $p = \frac{1}{2}$, P_1 is a uniform distribution on the closed interval $C := [0, \frac{1}{2}]$, and P_2 is a discrete distribution on $D := \{\frac{2}{3}, \frac{5}{6}, 1\}$. For this mixed distribution, in Subsection 2.6, we have determined the optimal sets of n -means and the n th quantization errors for all $n \geq 2$. We further showed that the quantization dimension of P exists, and equals the quantization dimension of P_1 , which again equals one, which is the dimension of the underlying space. For such a mixed distribution quantization coefficient also exists. In Section 3, for a mixed distribution $P := pP_1 + (1-p)P_2$, where P_1 is an absolutely continuous probability measure supported by the closed interval $C := [0, 1]$, and P_2 is discrete on $D := \{0, 1\}$, we mentioned a rule how to determine the optimal sets of n -means. In Proposition 3.2, for a special case, we gave a closed formula to determine the optimal sets of n -means and the n th quantization errors for all $n \geq 2$. In Remark 3.3, we proved a claim that the optimal sets for a mixed distribution may not be unique. In Section 4, we determined the optimal sets of n -means, and the n th quantization errors for all $n \geq 2$ for a mixed distribution $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$, where P_1 is a Cantor distribution with support lying in the closed interval $[0, \frac{1}{2}]$, and P_2 is discrete on $D := \{\frac{2}{3}, \frac{5}{6}, 1\}$. We further showed that the quantization dimension of this mixed distribution exists, but the quantization coefficient does not exist. In Section 5, we mentioned some open problems to be investigated on mixed distributions. In Section 6, we considered a mixed distribution $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$, where both P_1 and P_2 are Cantor distributions. For this mixed distribution, we determined the optimal sets of n -means and the n th quantization errors for all $n \geq 2$. Further we showed that the quantization dimension of this P exists, and satisfies $D(P) = \max\{D(P_1), D(P_2)\}$, but the quantization coefficient for P does not exist. Finally, we would like to mention that mixed distributions are an exciting new area for optimal quantization, and the results in this paper will give a motivation and insight into more general problems.

2. QUANTIZATION WITH P_1 UNIFORM AND P_2 DISCRETE

Let P_1 be a uniform distribution on the closed interval $C := [0, \frac{1}{2}]$, i.e., P_1 is a probability distribution on \mathbb{R} with probability density function g given by

$$g(x) = \begin{cases} 2 & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Let P_2 be a discrete probability distribution on \mathbb{R} with probability mass function h given by $h(x) = \frac{1}{3}$ for $x \in D$, and $h(x) = 0$ for $x \in \mathbb{R} \setminus D$, where $D := \{\frac{2}{3}, \frac{5}{6}, 1\}$. Let P be the mixed distribution on \mathbb{R} such that $P = \frac{1}{2}P_1 + \frac{1}{2}P_2$. Notice that the support of P_1 is C , and the support of P_2 is D implying that the support of P is $C \cup D$. Thus, for a Borel subset A of \mathbb{R} , we can write

$$P(A) = \frac{1}{2}P_1(A \cap C) + \frac{1}{2}P_2(A \cap D).$$

We now prove the following lemma.

Lemma 2.1. *Let $E(X)$ and $V := V(X)$ represent the expected value and the variance of a random variable X with distribution P . Then, $E(X) = \frac{13}{24}$ and $V = \frac{181}{1728} = 0.104745$.*

Proof. We have

$$\begin{aligned} E(X) &= \int x dP = \frac{1}{2} \int x dP_1 + \frac{1}{2} \int x dP_2 = \frac{1}{2} \int_{[0, \frac{1}{2}]} x g(x) dx + \frac{1}{2} \sum_{x \in D} x h(x) = \frac{13}{24}, \text{ and} \\ E(X^2) &= \int x^2 dP = \frac{1}{2} \int x^2 dP_1 + \frac{1}{2} \int x^2 dP_2 = \frac{1}{2} \int_{[0, \frac{1}{2}]} x^2 g(x) dx + \frac{1}{2} \sum_{x \in D} x^2 h(x) = \frac{43}{108}, \end{aligned}$$

implying $V := V(X) = E(X^2) - (E(X))^2 = \frac{43}{108} - (\frac{13}{24})^2 = \frac{181}{1728}$. Thus, the lemma is yielded. \square

Note 2.2. Following the standard rule of probability, we see that $E\|X - a\|^2 = \int (x - a)^2 dP = V(X) + (a - E(X))^2 = V + (a - \frac{13}{24})^2$, which yields the fact that the optimal set of one-mean consists of the expected value $\frac{13}{24}$, and the corresponding quantization error is the variance V of the random variable X . By $P(\cdot|C)$, we denote the restriction of the probability measure P on the interval C , i.e., $P(\cdot|C) = \frac{P(\cdot \cap C)}{P(C)}$, in other words, for any Borel subset B of C we have $P(B|C) = \frac{P(B \cap C)}{P(C)}$. Notice that $P(\cdot|C)$ is a uniform distribution with density function f given by

$$f(x) = \begin{cases} 2 & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases}$$

implying the fact that $P(\cdot|C) = P_1$. Similarly, $P(\cdot|D) = P_2$. In the sequel, for $n \in \mathbb{N}$ and $i = 1, 2$, by $\alpha_n(P_i)$ and $V_n(P_i)$, it is meant the optimal sets of n -means and the n th quantization error with respect to the probability distributions P_i . If nothing is mentioned within a parenthesis, i.e., by α_n and V_n , it is meant an optimal set of n -means and the n th quantization error with respect to the mixed distribution P .

Proposition 2.3. *Let P_1 be the uniform distribution on the closed interval $[a, b]$ and $n \in \mathbb{N}$. Then, the set $\{a + \frac{(2i-1)(b-a)}{2n} : 1 \leq i \leq n\}$ is a unique optimal set of n -means for P_1 , and the corresponding quantization error is given by $V_n(P_1) = \frac{(a-b)^2}{12n^2}$.*

Proof. Notice that the probability density function g of P_1 is given by

$$g(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

Since P_1 is uniformly distributed on $[a, b]$, the boundaries of the Voronoi regions of an optimal set of n -means will divide the interval $[a, b]$ into n equal subintervals, i.e., the boundaries of the

Voronoi regions are given by

$$\left\{ a, a + \frac{(b-a)}{n}, a + \frac{2(b-a)}{n}, \dots, a + \frac{(n-1)(b-a)}{n}, a + \frac{n(b-a)}{n} \right\}.$$

This implies that an optimal set of n -means for P_1 is unique, and it consists of the midpoints of the boundaries of the Voronoi regions, i.e., the optimal set of n -means for P_1 is given by $\alpha_n(P_1) := \{a + \frac{(2i-1)(b-a)}{2n} : 1 \leq i \leq n\}$ for any $n \geq 1$. Then, the n th quantization error for P_1 due to the set $\alpha_n(P_1)$ is given by

$$V_n(P_1) = n \int_{[a, a + \frac{b-a}{n}]} \left(x - \left(a + \frac{b-a}{2n}\right)\right)^2 dP_1 = n \int_{[0, \frac{1}{2n}]} \frac{1}{b-a} \left(x - \frac{1}{4n}\right)^2 dx = \frac{(a-b)^2}{12n^2},$$

which yields the proposition. \square

Corollary 2.4. *Let P_1 be the uniform distribution on the closed interval $[0, \frac{1}{2}]$ and $n \in \mathbb{N}$. Then, the set $\{\frac{2i-1}{4n} : 1 \leq i \leq n\}$ is a unique optimal set of n -means for P_1 , and the corresponding quantization error is given by $V_n(P_1) = \frac{1}{48n^2}$.*

Remark 2.5. Notice that if $\beta \subset \mathbb{R}$, then

$$\begin{aligned} \int \min_{b \in \beta} \|x - b\|^2 dP &= \frac{1}{2} \int_{[0, \frac{1}{2}]} \min_{b \in \beta} (x - b)^2 g(x) dx + \frac{1}{2} \sum_{x \in D} \min_{b \in \beta} (x - b)^2 h(x), \text{ and so,} \\ (1) \quad \int \min_{b \in \beta} \|x - b\|^2 dP &= \int_{[0, \frac{1}{2}]} \min_{b \in \beta} (x - b)^2 dx + \frac{1}{6} \sum_{x \in D} \min_{b \in \beta} (x - b)^2. \end{aligned}$$

2.6. Optimal sets of n -means and the errors for all $n \geq 2$. In this subsection, we first determine the optimal sets of n -means and the n th quantization error for the mixed distribution P . Then, we show that the quantization dimension of P exists and equals the quantization dimension of P_1 , which again equals one, which is the dimension of the underlying space. To determine the distortion error in this subsection we will frequently use equation (1).

Lemma 2.6.1. *Let α be an optimal set of two-means. Then, $\alpha = \{\frac{1}{4}, \frac{5}{6}\}$ with quantization error $V_2 = \frac{17}{864} = 0.0196759$.*

Proof. Consider the set of two-points β given by $\beta := \{\frac{1}{4}, \frac{5}{6}\}$. Then, the distortion error is

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \int_{[0, \frac{1}{2}]} \left(x - \frac{1}{4}\right)^2 dx + \frac{1}{6} \sum_{x \in D} \left(x - \frac{5}{6}\right)^2 = \frac{17}{864} = 0.0196759.$$

Since V_2 is the quantization error for two-means we have $V_2 \leq 0.0196759$. Let $\alpha := \{a_1, a_2\}$ be an optimal set of two-means with $a_1 < a_2$. Since the optimal points are the centroids of their own Voronoi regions, we have $0 < a_1 < a_1 < a_2 \leq 1$. If $\frac{13}{32} \leq a_1$, then

$$V_2 \geq \int_{[0, \frac{13}{32}]} \left(x - \frac{13}{32}\right)^2 dx = \frac{2197}{98304} = 0.022349 > V_2,$$

which is a contradiction. So, we can assume that $a_1 \leq \frac{13}{32}$. We now show that the Voronoi region of a_1 does not contain any point from D . For the sake of contradiction, assume that the Voronoi region of a_1 contains points from D . Then, the following two case can arise:

Case 1. $\frac{2}{3} \leq \frac{1}{2}(a_1 + a_2) < \frac{5}{6}$.

Then, $a_1 = E(X : X \in C \cup \{\frac{2}{3}\}) = \frac{17}{48}$ and $a_2 = E(X : X \in \{\frac{5}{6}, 1\}) = \frac{11}{12}$, and so $\frac{1}{2}(a_1 + a_2) = \frac{61}{96} < \frac{2}{3}$, which is a contradiction.

Case 2. $\frac{5}{6} \leq \frac{1}{2}(a_1 + a_2) < 1$.

Then, $a_1 = E(X : X \in C \cup \{\frac{2}{3}, \frac{5}{6}\}) = \frac{9}{20}$ and $a_2 = 1$, and so $\frac{1}{2}(a_1 + a_2) = \frac{29}{40} < \frac{5}{6}$, which is a contradiction.

By Case 1 and Case 2, we can assume that the Voronoi region of a_1 does not contain any point from D . We now show that the Voronoi region of a_2 does not contain any point from C . Suppose that the Voronoi region of a_2 contains points from C . Then, the distortion error is given by

$$\begin{aligned} & \int_{[0, \frac{1}{2}(a_1+a_2)]} (x - a_1)^2 dx + \int_{[\frac{1}{2}(a_1+a_2), \frac{1}{2}]} (x - a_2)^2 dx + \frac{1}{6} \sum_{x \in D} (x - a_2)^2 \\ &= \frac{1}{108} (27a_1^3 + 27a_1^2a_2 - 27a_1a_2^2 - 27a_2^3 + 108a_2^2 - 117a_2 + 43), \end{aligned}$$

which is minimum when $a_1 = \frac{5}{24}$ and $a_2 = \frac{19}{24}$, and the minimum value is $\frac{37}{1728} = 0.021412 > V_2$, which leads to a contradiction. So, we can assume that the Voronoi region of a_2 does not contain any point from C . Thus, we have $a_1 = \frac{1}{4}$ and $a_2 = \frac{5}{6}$, and the corresponding quantization error is $V_2 = \frac{17}{864} = 0.0196759$. This, completes the proof of the lemma. \square

Lemma 2.6.2. *Let α be an optimal set of three-means. Then, $\alpha = \{0.191074, 0.573223, \frac{11}{12}\}$ with quantization error $V_3 = 0.0106152$.*

Proof. Let us consider the set of three-points $\beta := \{0.191074, 0.573223, \frac{11}{12}\}$. Since $0.382149 = \frac{1}{2}(0.191074 + 0.573223) < \frac{1}{2} < \frac{2}{3} < \frac{1}{2}(0.573223 + \frac{11}{12}) = 0.744945 < \frac{5}{6}$, the distortion error due to the set β is given by

$$\begin{aligned} & \int \min_{b \in \beta} \|x - b\|^2 dP = \int_{[0, 0.382149]} (x - 0.191074)^2 dx + \int_{[0.382149, \frac{1}{2}]} (x - 0.573223)^2 dx \\ &+ \frac{1}{6} \left(\frac{2}{3} - 0.573223 \right)^2 + \frac{1}{6} \left(\frac{5}{6} - \frac{11}{12} \right)^2 + \frac{1}{6} \left(1 - \frac{11}{12} \right)^2 = 0.0106152. \end{aligned}$$

Since V_3 is the quantization error for three-means, we have $V_3 \leq 0.0106152$. Let $\alpha := \{a_1, a_2, a_3\}$ be an optimal set of three-means with $a_1 < a_2 < a_3$. Since the optimal points are the centroids of their own Voronoi regions, we have $0 < a_1 < a_2 < a_3 \leq 1$. If $\frac{3}{8} \leq a_1$, then

$$V_3 \geq \int_{[0, \frac{3}{8}]} (x - \frac{3}{8})^2 dx = \frac{9}{512} = 0.0175781 > V_3,$$

which leads to a contradiction. So, we can assume that $a_1 < \frac{3}{8}$. If the Voronoi region of a_2 does not contain any point from C , then as the points of D are equidistant from each other with equal probability, we will have either $(a_2 = \frac{1}{2}(\frac{2}{3} + \frac{5}{6}) = \frac{3}{4}$ and $a_3 = 1)$, or $(a_2 = \frac{2}{3}$ and $a_3 = \frac{1}{2}(\frac{5}{6} + 1) = \frac{11}{12})$. In any case, the distortion error is

$$\int_{[0, \frac{1}{2}]} (x - \frac{1}{4})^2 dx + \frac{1}{6} \left(\left(\frac{2}{3} - \frac{3}{4} \right)^2 + \left(\frac{5}{6} - \frac{3}{4} \right)^2 \right) = \frac{11}{864} = 0.0127315 > V_3,$$

which is a contradiction. So, we can assume that the Voronoi region of a_2 contains points from C . If the Voronoi region of a_2 does not contain any point from D , we must have $a_1 = \frac{1}{8}$, $a_2 = \frac{3}{8}$, and $a_3 = \frac{5}{6}$. Then, the distortion error is

$$\int_{[0, \frac{1}{4}]} (x - \frac{1}{8})^2 dx + \int_{[\frac{1}{4}, \frac{1}{2}]} (x - \frac{3}{8})^2 dx + \frac{1}{6} \left(\left(\frac{2}{3} - \frac{5}{6} \right)^2 + \left(1 - \frac{5}{6} \right)^2 \right) = \frac{41}{3456} = 0.0118634 > V_3,$$

which leads to a contradiction. Therefore, we can assume that the Voronoi region of a_2 contains points from C as well as from D . We now show that the Voronoi region of a_2 contains only the point $\frac{2}{3}$ from D . On the contrary, assume that the Voronoi region of a_2 contains the points $\frac{2}{3}$ and $\frac{5}{6}$ from D . Then, we must have $a_3 = 1$, and so the distortion error is

$$\begin{aligned} & \int_{[0, \frac{a_1+a_2}{2}]} (x - a_1)^2 dx + \int_{[\frac{a_1+a_2}{2}, \frac{1}{2}]} (x - a_2)^2 dx + \frac{1}{6} \left(\left(\frac{2}{3} - a_2 \right)^2 + \left(\frac{5}{6} - a_2 \right)^2 \right) \\ &= \frac{1}{108} (27a_1^3 + 27a_1^2a_2 - 27a_1a_2^2 - 27a_2^3 + 90a_2^2 - 81a_2 + 25), \end{aligned}$$

which is minimum when $a_1 = \frac{1}{4}$ and $a_2 = \frac{3}{4}$, and the minimum value is $\frac{11}{864} = 0.0127315 > V_3$, which is a contradiction. Therefore, the Voronoi region of a_2 contains only the point $\frac{2}{3}$ from D . This implies $a_3 = \frac{1}{2}(\frac{5}{6} + 1) = \frac{11}{12}$, and then the distortion error is

$$\begin{aligned} & \int_{[0, \frac{a_1+a_2}{2}]} (x - a_1)^2 dx + \int_{[\frac{a_1+a_2}{2}, \frac{1}{2}]} (x - a_2)^2 dx + \frac{1}{6}(\frac{2}{3} - a_2)^2 + \frac{1}{6} \left((1 - \frac{11}{12})^2 + (\frac{5}{6} - \frac{11}{12})^2 \right) \\ &= \frac{1}{144} (36a_1^3 + 36a_1^2a_2 - 36a_1a_2^2 - 36a_2^3 + 96a_2^2 - 68a_2 + 17), \end{aligned}$$

which is minimum when $a_1 = 0.191074$ and $a_2 = 0.573223$, and the corresponding distortion error is $V_3 = 0.0106152$. Moreover, we have seen $a_3 = \frac{11}{12}$. Thus, the proof of the lemma is complete. \square

Lemma 2.6.3. *Let α be an optimal set of four-means. Then, $\alpha = \{\frac{1}{4}, \frac{3}{8}, \frac{3}{4}, 1\}$, or $\alpha = \{\frac{1}{4}, \frac{3}{8}, \frac{2}{3}, \frac{11}{12}\}$, and the quantization error is $V_4 = \frac{17}{3456} = 0.00491898$.*

Proof. Let us consider the set of four-points $\beta := \{\frac{1}{4}, \frac{3}{8}, \frac{3}{4}, 1\}$. Then, the distortion error due to the set β is

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \int_{[0, \frac{1}{4}]} (x - \frac{1}{8})^2 dx + \int_{[\frac{1}{4}, \frac{1}{2}]} (x - \frac{3}{8})^2 dx + \frac{1}{6} \left((\frac{2}{3} - \frac{3}{4})^2 + (\frac{5}{6} - \frac{3}{4})^2 \right) = \frac{17}{3456}.$$

Since V_4 is the quantization error for four-means, we have $V_4 \leq \frac{17}{3456} = 0.00491898$. Let $\alpha := \{a_1 < a_2 < a_3 < a_4\}$ be an optimal set of four-means. Since the optimal points are the centroids of their own Voronoi regions, we have $0 < a_1 < \dots < a_4 \leq 1$. If the Voronoi region of a_2 does not contain points from C , then

$$V_4 \geq \int_{[0, \frac{1}{2}]} (x - \frac{1}{4})^2 dx = \frac{1}{96} = 0.0104167 > V_4,$$

which gives a contradiction, and so, we can assume that the Voronoi region of a_2 contains points from C . If the Voronoi region of a_2 contains points from D , then it can contain only the point $\frac{2}{3}$ from D , and in that case $a_3 = \frac{5}{6}$ and $a_4 = 1$, which leads to the distortion error as

$$\begin{aligned} & \int_{[0, \frac{a_1+a_2}{2}]} (x - a_1)^2 dx + \int_{[\frac{a_1+a_2}{2}, \frac{1}{2}]} (x - a_2)^2 dx + \frac{1}{6}(\frac{2}{3} - a_2)^2 \\ &= \frac{1}{216} (54a_1^3 + 54a_1^2a_2 - 54a_1a_2^2 - 54a_2^3 + 144a_2^2 - 102a_2 + 25), \end{aligned}$$

which is minimum when $a_1 = 0.191074$ and $a_2 = 0.573223$, and then, the minimum value is $0.00830043 > V_4$, which is a contradiction. So, the Voronoi region of a_2 does not contain any point from D . If the Voronoi region of a_3 does not contain any point from D , then $a_4 = \frac{5}{6}$ yielding

$$V_4 \geq \frac{1}{6} \left((\frac{2}{3} - \frac{5}{6})^2 + (1 - \frac{5}{6})^2 \right) = \frac{1}{108} = 0.00925926 > V_4,$$

which leads to a contradiction. So, the Voronoi region of a_3 contains at least one point from D . Suppose that the Voronoi region of a_3 contains points from C as well. Then, the following two cases can arise:

Case 1. $\frac{2}{3} \in M(a_3|\alpha)$.

Then, $a_4 = \frac{11}{12}$, and the distortion error is

$$\begin{aligned} & \int_{[0, \frac{a_1+a_2}{2}]} (x - a_1)^2 dx + \int_{[\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]} (x - a_2)^2 dx + \int_{[\frac{a_2+a_3}{2}, \frac{1}{2}]} (x - a_3)^2 dx + \frac{1}{6}(\frac{2}{3} - a_3)^2 \\ & \quad + \frac{1}{6} \left((1 - \frac{11}{12})^2 + (\frac{5}{6} - \frac{11}{12})^2 \right) \\ &= \frac{1}{144} (36a_1^3 + 36a_1^2a_2 - 36a_1a_2^2 + 4(9a_2^2 - 17)a_3 + (96 - 36a_2)a_3^2 - 36a_3^3 + 17) \end{aligned}$$

which is minimum if $a_1 = 0.118238$, $a_2 = 0.354715$, and $a_3 = 0.645285$, and the minimum value is $0.00506623 > V_4$, which is a contradiction.

Case 2. $\{\frac{2}{3}, \frac{5}{6}\} \subset M(a_3|\alpha)$.

Then, $a_4 = 1$, and the corresponding distortion error is

$$\begin{aligned} & \int_{[0, \frac{a_1+a_2}{2}]} (x - a_1)^2 dx + \int_{[\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]} (x - a_2)^2 dx + \int_{[\frac{a_2+a_3}{2}, \frac{1}{2}]} (x - a_3)^2 dx \\ & + \frac{1}{6} \left(\left(\frac{2}{3} - a_3 \right)^2 + \left(\frac{5}{6} - a_3 \right)^2 \right) \\ & = \frac{1}{108} \left(27a_1^3 + 27a_1^2a_2 - 27a_1a_2^2 + 27(a_2^2 - 3)a_3 + (90 - 27a_2)a_3^2 - 27a_3^3 + 25 \right), \end{aligned}$$

which is minimum if $a_1 = 0.0990219$, $a_2 = 0.297066$, and $a_3 = 0.702934$, and the minimum value is $0.00680992 > V_4$, which gives a contradiction.

By Case 1 and Case 2, we can assume that the Voronoi region of a_3 does not contain any point from C . Thus, we have $(a_1 = \frac{1}{4}, a_2 = \frac{3}{8}, a_3 = \frac{3}{4}, \text{ and } a_4 = 1)$, or $(a_1 = \frac{1}{4}, a_2 = \frac{3}{8}, a_3 = \frac{2}{3}, \text{ and } a_4 = \frac{11}{12})$, and the corresponding quantization error is $V_4 = \frac{17}{3456} = 0.00491898$. \square

Lemma 2.6.4. *Let α be an optimal set of five-means. Then, $\alpha = \{\frac{1}{8}, \frac{3}{8}, \frac{2}{3}, \frac{5}{6}, 1\}$, and the corresponding quantization error is $V_5 = \frac{1}{384} = 0.00260417$.*

Proof. Consider the set of five points $\beta := \{\frac{1}{4}, \frac{3}{8}, \frac{2}{3}, \frac{5}{6}, 1\}$. The distortion error due to the set β is given by

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \int_{[0, \frac{1}{4}]} (x - \frac{1}{8})^2 dx + \int_{[\frac{1}{4}, \frac{1}{2}]} (x - \frac{3}{8})^2 dx = \frac{1}{384} = 0.00260417.$$

Since V_5 is the quantization error for five-means, we have $V_5 \leq 0.00260417$. Let $\alpha := \{a_1 < a_2 < a_3 < a_4 < a_5\}$ be an optimal set of five-means. Since the optimal points are the centroids of their own Voronoi regions, we have $0 < a_1 < \dots < a_5 \leq 1$. If the Voronoi region of a_3 does not contain any point from D , then we must have $(a_1 = \frac{1}{12}, a_2 = \frac{1}{4}, a_3 = \frac{5}{12}, a_4 = \frac{3}{4}, \text{ and } a_5 = 1)$, or $(a_1 = \frac{1}{12}, a_2 = \frac{1}{4}, a_3 = \frac{5}{12}, a_4 = \frac{2}{3}, \text{ and } a_5 = \frac{11}{12})$ yielding the distortion error

$$3 \int_{[0, \frac{1}{6}]} (x - \frac{1}{12})^2 dx + \frac{1}{6} \left(\left(\frac{2}{3} - \frac{3}{4} \right)^2 + \left(\frac{5}{6} - \frac{3}{4} \right)^2 \right) = \frac{1}{288} = 0.00347222 > V_5,$$

which is a contradiction. So, we can assume that the Voronoi region of a_3 contains a point from D . In that case, we must have $a_4 = \frac{5}{6}$ and $a_5 = 1$. Suppose that the Voronoi region of a_3 contains points from C as well. Then, the distortion error is

$$\begin{aligned} & \int_{[0, \frac{a_1+a_2}{2}]} (x - a_1)^2 dx + \int_{[\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}]} (x - a_2)^2 dx + \int_{[\frac{a_2+a_3}{2}, \frac{1}{2}]} (x - a_3)^2 dx + \frac{1}{6} \left(\frac{2}{3} - a_3 \right)^2 \\ & = \frac{1}{216} (54a_1^3 + 54a_1^2a_2 - 54a_1a_2^2 + 6(9a_2^2 - 17)a_3 - 18(3a_2 - 8)a_3^2 - 54a_3^3 + 25), \end{aligned}$$

which is minimum if $a_1 = 0.118238$, $a_2 = 0.354715$, and $a_3 = 0.645285$, and the minimum value is $0.00275142 > V_5$, which is a contradiction. So, the Voronoi region of a_3 does not contain any point from C yielding $a_1 = \frac{1}{8}, a_2 = \frac{3}{8}, a_3 = \frac{2}{3}, a_4 = \frac{5}{6}$ and $a_5 = 1$, and the corresponding quantization error is $V_5 = \frac{1}{384} = 0.00260417$. Thus, the proof of the lemma is complete. \square

Theorem 2.6.5. *Let $n \in \mathbb{N}$ and $n \geq 5$, and let α_n be an optimal set of n -means for P and $\alpha_n(P_1)$ be the optimal set of n -means with respect to P_1 . Then,*

$$\alpha_n(P) = \alpha_{n-3}(P_1) \cup D, \text{ and } V_n(P) = \frac{1}{2}V_{n-3}(P_1).$$

Proof. If $n = 5$, by Lemma 2.6.4, we have $\alpha_5(P) = \{\frac{1}{8}, \frac{3}{8}, \frac{2}{3}, \frac{5}{6}, 1\}$ and $V_5(P) = \frac{1}{384}$, which by Corollary 2.4 yields that $\alpha_5(P) = \alpha_2(P_1) \cup D$ and $V_5(P) = \frac{1}{2}V_2(P_1)$, i.e., the theorem is true for $n = 5$. Proceeding in the similar way, as Lemma 2.6.4, we can show that the theorem is true

for $n = 6$ and $n = 7$. We now show that the theorem is true for all $n \geq 8$. Consider the set of eight points $\beta := \{\frac{1}{20}, \frac{3}{20}, \frac{1}{4}, \frac{7}{20}, \frac{9}{20}, \frac{2}{3}, \frac{5}{6}, 1\}$. The distortion error due to set β is given by

$$\int \min_{b \in \beta} \|x - b\|^2 dP = 5 \int_{[0, \frac{1}{10}]} (x - \frac{1}{20})^2 dx = \frac{1}{2400} = 0.000416667.$$

Since V_n is the n th quantization error for n -means for $n \geq 8$, we have $V_n \leq V_8 \leq 0.000416667$. Let $\alpha_n := \{a_1 < a_2 < \dots < a_n\}$ be an optimal set of n -means for $n \geq 8$, where $0 < a_1 < \dots < a_n \leq 1$. To prove the first part of the theorem, it is enough to show that $M(a_{n-2}|\alpha_n)$ does not contain any point from C , and $M(a_{n-3}|\alpha_n)$ does not contain any point from D . If $M(a_{n-2}|\alpha_n)$ does not contain any point from D , then

$$V_n \geq \frac{1}{6} \left(\left(\frac{2}{3} - \frac{3}{4} \right)^2 + \left(\frac{5}{6} - \frac{3}{4} \right)^2 \right) = \frac{1}{432} = 0.00231481 > V_n,$$

which leads to a contradiction. So, $M(a_{n-2}|\alpha_n)$ contains a point, in fact the point $\frac{2}{3}$, from D . If $M(a_{n-2}|\alpha_n)$ does not contain points from C , then $a_{n-2} = \frac{2}{3}$. Suppose that $M(a_{n-2}|\alpha_n)$ contains points from C . Then, $\frac{2}{3} \leq \frac{1}{2}(a_{n-2} + a_{n-1})$ implies $a_{n-2} \geq \frac{4}{3} - a_{n-1} = \frac{4}{3} - \frac{5}{6} = \frac{1}{2}$. The following three cases can arise:

Case 1. $\frac{1}{2} \leq a_{n-2} \leq \frac{7}{12}$.

Then, $V_n \geq \frac{1}{6} \left(\left(\frac{2}{3} - \frac{7}{12} \right)^2 \right) = \frac{1}{864} = 0.00115741 > V_n$, which is a contradiction.

Case 2. $\frac{7}{12} \leq a_{n-2} \leq \frac{5}{8}$.

Then, $\frac{1}{2}(a_{n-3} + a_{n-2}) < \frac{1}{2}$ implying $a_{n-3} < 1 - a_{n-2} \leq 1 - \frac{7}{12} = \frac{5}{12}$, and so

$$V_n \geq \int_{[\frac{5}{12}, \frac{1}{2}]} \left(x - \frac{5}{12} \right)^2 dx + \frac{1}{6} \left(\frac{2}{3} - \frac{5}{8} \right)^2 = \frac{5}{10368} = 0.000482253 > V_n,$$

which leads to a contradiction.

Case 3. $\frac{5}{8} \leq a_{n-2}$.

Then, $\frac{1}{2}(a_{n-3} + a_{n-2}) < \frac{1}{2}$ implying $a_{n-3} < 1 - a_{n-2} \leq 1 - \frac{5}{8} = \frac{3}{8}$, and so

$$V_n \geq \int_{[\frac{3}{8}, \frac{1}{2}]} \left(x - \frac{3}{8} \right)^2 dx = \frac{1}{1536} = 0.000651042 > V_n,$$

which gives contradiction.

Thus, in each case we arrive at a contradiction yielding the fact that $M(a_{n-2}|\alpha_n)$ does not contain any point from C . If $M(a_{n-3}|\alpha_n)$ contains any point from D , say $\frac{2}{3}$, then we will have

$$M(a_{n-2}|\alpha) \cup M(a_{n-1}|\alpha) \cup M(a_n|\alpha) = \left\{ \frac{5}{6}, 1 \right\},$$

which by Proposition 1.1 implies that either $(a_{n-2} = a_{n-1} = \frac{5}{6}, \text{ and } a_n = 1)$, or $(a_{n-2} = \frac{5}{6}, \text{ and } a_{n-1} = a_n = 1)$, which contradicts the fact that $0 < a_1 < \dots < a_{n-2} < a_{n-1} < a_n \leq 1$. Thus, $M(a_{n-3}|\alpha)$ does not contain any point from D . Hence, $\alpha_n(P) = \alpha_{n-3}(P_1) \cup D$, and so,

$$V_n(P) = \int_C \min_{a \in \alpha_{n-3}(P_1)} (x - a)^2 dx + \frac{1}{6} \sum_{x \in D} \min_{a \in D} (x - a)^2 = \frac{1}{2} \int_C \min_{a \in \alpha_{n-3}(P_1)} (x - a)^2 2dx$$

implying $V_n(P) = \frac{1}{2} V_{n-3}(P_1)$. Thus, the proof of the theorem is complete. \square

Proposition 2.6.6. *Let P be the mixed distribution as defined before. Then,*

$$\lim_{n \rightarrow \infty} n^2 V_n(P) = \frac{1}{96}.$$

Proof. By Corollary 2.4 and Theorem 2.6.5, we have

$$\lim_{n \rightarrow \infty} n^2 V_n(P) = \frac{1}{2} \lim_{n \rightarrow \infty} n^2 V_{n-3}(P_1) = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n^2}{48(n-3)^2} = \frac{1}{96},$$

and thus, the proposition is yielded. \square

Remark 2.6.7. By Proposition 2.6.6, it follows that $\lim_{n \rightarrow \infty} n^2 V_n(P) = \frac{1}{96}$, i.e., one-dimensional quantization coefficient for the mixed distribution P is finite and positive implying the fact that the quantization dimension of the mixed distribution P exists, and equals one, which is the dimension of the underlying space. It is known that for a probability measure P on \mathbb{R}^d with non-vanishing absolutely continuous part $\lim_{n \rightarrow \infty} n^{\frac{2}{d}} V_n(P)$ is finite and strictly positive, i.e., the quantization dimension of P exists, and equals the dimension d of the underlying space (see [BW]). Thus, for the mixed distribution P considered in this section, we see that $D(P) = D(P_1) = 1$.

3. A RULE TO DETERMINE OPTIMAL QUANTIZERS

Let $0 < p < 1$ be fixed. Let P be a mixed distribution given by $P = pP_1 + (1-p)P_2$ with the support of P_1 equals C and the support of P_2 equals D , such that P_1 is continuous on C , and P_2 is discrete on D , and $D \subset C$. It is well-known that the optimal set of one-mean consists of the expected value and the corresponding quantization error is the variance V of the P -distributed random variable X . Assume that P_1 is absolutely continuous on $C := [0, 1]$, and P_2 is discrete on $D := \{0, 1\}$. Then, in the following note we give a rule how to obtain the optimal sets of n -means for the mixed distribution P for any $n \geq 2$.

Note 3.1. Let $\alpha_n := \{a_1, a_2, \dots, a_n\}$ be an optimal set of n -means for P such that $0 \leq a_1 < a_2 < \dots < a_n \leq 1$. Write

$$(2) \quad M(a_i | \alpha_n) := \begin{cases} [0, \frac{a_1+a_2}{2}] & \text{if } i = 1, \\ [\frac{a_{i-1}+a_i}{2}, \frac{a_i+a_{i+1}}{2}] & \text{if } 2 \leq i \leq n-1, \\ [\frac{a_{n-1}+a_n}{2}, 1] & \text{if } i = n, \end{cases}$$

where $M(a_i | \alpha)$ represent the Voronoi regions of a_i for all $1 \leq i \leq n$ with respect to the set α_n . Since the optimal points are the centroids of their own Voronoi regions, we have $a_i = E(X : X \in M(a_i | \alpha))$ for all $1 \leq i \leq n$. Solving the n equations one can obtain the optimal sets of n -means for the mixed distribution P . Once, an optimal set of n -means is known, the corresponding quantization error can easily be determined.

Let us now give the following proposition.

Proposition 3.2. Let α_n be an optimal set of n -means and V_n is the corresponding quantization error for $n \geq 2$ for the mixed distribution $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$ such that P_1 is uniformly distributed on $C := [0, 1]$ with probability density function g given by

$$g(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases}$$

and P_2 is discrete on $D := \{1\}$ with mass function h given by $h(1) = 1$. Then, for $n \geq 2$,

$$\alpha_n := \left\{ \frac{(2i-1)(-\sqrt{n^2-n+1}+2n-1)}{2(n-1)n} : 1 \leq i \leq n \right\}$$

$$\text{and } V_n = \frac{4n^2 - 4(\sqrt{n^2-n+1}+1)n + 2\sqrt{n^2-n+1}+7}{12(\sqrt{n^2-n+1}+2n-1)^2}.$$

Proof. As mentioned in Note 3.1, solving the n equations $a_i = E(X : X \in M(a_i | \alpha))$, we obtain

$$a_i = \frac{(2i-1)(-\sqrt{n^2-n+1}+2n-1)}{2(n-1)n},$$

for all $1 \leq i \leq n$, and hence, the corresponding quantization error is given by

$$V_n = \int_0^{\frac{1}{2}(a_1+a_2)} (x-a_1)^2 dx + \sum_{i=2}^{n-1} \int_{\frac{1}{2}(a_{i-1}+a_i)}^{\frac{1}{2}(a_i+a_{i+1})} (x-a_i)^2 dx + \int_{\frac{1}{2}(a_{n-1}+a_n)}^1 (x-a_n)^2 dx + \frac{1}{2}(a_n-1)^2,$$

which upon simplification yields $V_n = \frac{4n^2 - 4(\sqrt{n^2 - n + 1} + 1)n + 2\sqrt{n^2 - n + 1} + 7}{12(\sqrt{n^2 - n + 1} + 2n - 1)^2}$. Thus, the proof of the proposition is complete. \square

Remark 3.3. Let P_1 be absolutely continuous on $C := [0, 1]$ and P_2 be discrete on D with $D \subset C$. Then, if $D := \{0, 1\}$, the system of equations in (3) has a unique solution implying that there exists a unique optimal set of n -means for the mixed distribution $P := pP_1 + (1 - p)P_2$ for each $n \in \mathbb{N}$. If $D \cap \text{Int}(C)$ is nonempty, where $\text{Int}(C)$ represents the interior of C , then the optimal sets of n -means for the mixed distribution P for all $n \in \mathbb{N}$ is not necessarily unique, see Proposition 3.4.

Proposition 3.4. Let $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$, where P_1 is uniformly distributed on $C := [0, 1]$ and P_2 is discrete on $D := \{\frac{1}{2}\}$. Then, P has two different optimal sets of two-means.

Proof. Let $\alpha := \{a_1, a_2\}$ be an optimal set of two means for P with $0 < a_1 < a_2 < 1$. Then, P -almost surely, we have $C = M(a_1|\alpha) \cup M(a_2|\alpha)$ implying that either $\frac{1}{2} \in M(a_1|\alpha)$, or $\frac{1}{2} \in M(a_2|\alpha)$. First, assume that $\frac{1}{2} \in M(a_1|\alpha)$, i.e., $0 < a_1 < \frac{1}{2} \leq \frac{1}{2}(a_1 + a_2)$. Then,

$$a_1 = E(X : X \in [0, \frac{1}{2}(a_1 + a_2)]) = \frac{\int_0^{\frac{a+b}{2}} x dx + \frac{1}{2}}{\int_0^{\frac{a+b}{2}} 1 dx + 1} = \frac{a^2 + 2ab + b^2 + 4}{4(a + b + 2)}, \text{ and}$$

$$a_2 = E(X : X \in [\frac{1}{2}(a_1 + a_2), 1]) = \frac{\int_{\frac{a+b}{2}}^1 x dx}{\int_{\frac{a+b}{2}}^1 1 dx} = \frac{1}{4}(a + b + 2).$$

Solving the above two equations, we have $a_1 = \frac{1}{4}(-5 + 3\sqrt{5})$ and $a_2 = \frac{1}{4}(1 + \sqrt{5})$, and the corresponding quantization error is given by

$$V_2(P) = \int \min_{a \in \alpha} \|x - a\|^2 dP = \frac{1}{2} \int \min_{a \in \alpha} (x - a)^2 dP_1 + \frac{1}{2} \int \min_{a \in \alpha} (x - a)^2 dP_2$$

$$= \frac{1}{2} \int_0^{\frac{a_1+a_2}{2}} (x - a_1)^2 dx + \frac{1}{2} \int_{\frac{a_1+a_2}{2}}^1 (x - a_2)^2 dx + \frac{1}{2} \left(\frac{1}{2} - a_1 \right)^2 = 0.0191242.$$

Next, assume that $\frac{1}{2} \in M(a_2|\alpha)$, i.e., $\frac{1}{2}(a_1 + a_2) \leq \frac{1}{2} < a_2 < 1$. Then,

$$a_1 = E(X : X \in [0, \frac{1}{2}(a_1 + a_2)]) = \frac{\int_0^{\frac{a+b}{2}} x dx}{\int_0^{\frac{a+b}{2}} 1 dx} = \frac{a + b}{4}, \text{ and}$$

$$a_2 = E(X : X \in [\frac{1}{2}(a_1 + a_2), 1]) = \frac{\int_{\frac{a+b}{2}}^1 1x dx + \frac{1}{2}}{\int_{\frac{a+b}{2}}^1 1 dx + 1} = \frac{a^2 + 2ab + b^2 - 8}{4(a + b - 4)}.$$

Solving the above two equations, we have $a_1 = \frac{1}{4}(3 - \sqrt{5})$ and $a_2 = \frac{3}{4}(3 - \sqrt{5})$, and as before, the corresponding quantization error is give by

$$V_2(P) = \frac{1}{2} \int_0^{\frac{a_1+a_2}{2}} (x - a_1)^2 dx + \frac{1}{2} \int_{\frac{a_1+a_2}{2}}^1 (x - a_2)^2 dx + \frac{1}{2} \left(\frac{1}{2} - a_2 \right)^2 = 0.0191242.$$

Thus, we see that there are two different optimal sets of two-means with same quantization error, which is the proposition. \square

Remark 3.5. For each even positive integer n , for the mixed distribution $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$ given by Proposition 3.4, there are two different optimal sets of n -means, and between the two different optimal sets of n -means, one is the reflection of the other with respect to the point $\frac{1}{2}$.

4. QUANTIZATION WITH P_1 A CANTOR DISTRIBUTION AND P_2 DISCRETE

In this section, we consider a mixed distribution $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$, where P_1 is a Cantor distribution given by $P_1 = \frac{1}{2}P_1 \circ S_1^{-1} + \frac{1}{2}P_1 \circ S_2^{-1}$, where $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{1}{3}$ for all $x \in \mathbb{R}$, and P_2 is a discrete distribution on $D := \{\frac{2}{3}, \frac{5}{6}, 1\}$ with density function h given by $h(x) = \frac{1}{3}$ for all $x \in D$. By a *word*, or a *string* of length k over the alphabet $\{1, 2\}$, it is meant $\sigma := \sigma_1\sigma_2 \cdots \sigma_k$, where $\sigma_j \in \{1, 2\}$ for $1 \leq j \leq k$. A word of length zero is called the empty word and is denoted by \emptyset . Length of a word σ is denoted by $|\sigma|$. The set of all words over the alphabet $\{1, 2\}$ including the empty word \emptyset is denoted by $\{1, 2\}^*$. For two words $\sigma := \sigma_1\sigma_2 \cdots \sigma_{|\sigma|}$ and $\tau := \tau_1\tau_2 \cdots \tau_{|\tau|}$, by $\sigma\tau$, it is meant the concatenation of the words σ and τ . If $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$, we write $S_\sigma := S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_k}$, and $J_\sigma = S_\sigma(J)$, where $J = J_\emptyset := [0, \frac{1}{2}]$. S_1 and S_2 generate the Cantor set $C := \bigcap_{k \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2\}^k} J_\sigma$. C is the support of the probability distribution P_1 . Notice that the support of the Mixed distribution P is $C \cup D$. For any $\sigma \in \{1, 2\}^k$, $k \geq 1$, the intervals $J_{\sigma 1}$ and $J_{\sigma 2}$ into which J_σ is split up at the $(k+1)$ th level are called the *children* of J_σ .

The following lemma is well-known and appears in many places, for example, see [GL2, R1].

Lemma 4.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable and $k \in \mathbb{N}$. Then*

$$\int f dP = \sum_{\sigma \in \{1, 2\}^k} \frac{1}{2^k} \int f \circ S_\sigma dP.$$

Lemma 4.2. *Let X_1 be a P_1 -distributed random variable. Then, its expectation and the variance are respectively give by $E(X_1) = \frac{1}{4}$ and $V(X_1) = \frac{1}{32}$, and for any $x_0 \in \mathbb{R}$, $\int (x - x_0)^2 dP_1(x) = V(X_1) + (x_0 - \frac{1}{4})^2$.*

Proof. Using Lemma 4.1, we have $E(X_1) = \int x dP_1 = \frac{1}{2} \int \frac{1}{3}x dP_1 + \frac{1}{2} \int (\frac{1}{3}x + \frac{1}{3}) dP_1 = \frac{1}{6} E(X_1) + \frac{1}{6} E(X_1) + \frac{1}{6}$ implying $E(X_1) = \frac{1}{4}$. Again,

$$E(X_1^2) = \int x^2 dP_1 = \frac{1}{2} \int \frac{1}{9} x^2 dP_1 + \frac{1}{2} \int \left(\frac{1}{3}x + \frac{1}{3}\right)^2 dP_1 = \frac{1}{9} E(X_1^2) + \frac{1}{9} E(X_1) + \frac{1}{18},$$

which yields $E(X_1^2) = \frac{3}{32}$, and hence $V(X_1) = E(X_1 - E(X_1))^2 = E(X_1^2) - (E(X_1))^2 = \frac{3}{32} - (\frac{1}{4})^2 = \frac{1}{32}$. Then, following the standard theory of probability, we have $\int (x - x_0)^2 dP_1 = V(X_1) + (x_0 - E(X_1))^2$, and thus the lemma is yielded. \square

Definition 4.3. For $n \in \mathbb{N}$ with $n \geq 2$, let $\ell(n)$ be the unique natural number with $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. For $I \subset \{1, 2\}^{\ell(n)}$ with $\text{card}(I) = n - 2^{\ell(n)}$ let $\beta_n(I)$ be the set consisting of all midpoints $a(\sigma)$ of intervals J_σ with $\sigma \in \{1, 2\}^{\ell(n)} \setminus I$ and all midpoints $a(\sigma 1)$, $a(\sigma 2)$ of the children of J_σ with $\sigma \in I$, i.e.,

$$\beta_n(I) = \{a(\sigma) : \sigma \in \{1, 2\}^{\ell(n)} \setminus I\} \cup \{a(\sigma 1) : \sigma \in I\} \cup \{a(\sigma 2) : \sigma \in I\}.$$

The following proposition follows due to [GL2, Definition 3.5 and Proposition 3.7].

Proposition 4.4. *Let $\beta_n(I)$ be the set for $n \geq 2$ given by Definition 4.3. Then, $\beta_n(I)$ forms an optimal set of n -means for P_1 , and the corresponding quantization error is given by*

$$V_n(P_1) = \int \min_{a \in \beta_n(I)} \|x - a\|^2 dP_1 = \frac{1}{18^{\ell(n)}} \cdot \frac{1}{32} \left(2^{\ell(n)+1} - n + \frac{1}{9} (n - 2^{\ell(n)}) \right).$$

Lemma 4.5. *Let $E(X)$ and $V := V(X)$ represent the expected value and the variance of a random variable X with distribution P . Then, $E(X) = \frac{13}{24}$ and $V = \frac{95}{864} = 0.109954$.*

Proof. In this proof we use the results from Lemma 4.1. We have

$$E(X) = \int x dP = \frac{1}{2} \int x dP_1 + \frac{1}{2} \int x dP_2 = \frac{1}{2} \int x dP_1 + \frac{1}{2} \sum_{x \in D} x h(x) = \frac{13}{24}, \text{ and}$$

$$E(X^2) = \int x^2 dP = \frac{1}{2} \int x^2 dP_1 + \frac{1}{2} \sum_{x \in D} x^2 h(x) = \frac{697}{1728},$$

implying $V := V(X) = E(X^2) - (E(X))^2 = \frac{697}{1728} - \left(\frac{13}{24}\right)^2 = \frac{95}{864}$. Thus, the lemma is yielded. \square

Note 4.6. Since $E\|X - a\|^2 = \int (x - a)^2 dP = V(X) + (a - E(X))^2 = V + (a - \frac{13}{24})^2$, it follows that the optimal set of one-mean for the mixed distribution P consists of the expected value $\frac{13}{24}$, and the corresponding quantization error is the variance V of the random variable X . For any $\sigma \in \{1, 2\}^*$, by $a(\sigma)$, it is meant $a(\sigma) := E(X_1 : X_1 \in J_\sigma)$, where X_1 is a P_1 distributed random variable, i.e., $a(\sigma) = S_\sigma(\frac{1}{4})$. Notice that for any $\sigma \in \{1, 2\}^*$, and for any $x_0 \in \mathbb{R}$, we have

$$(3) \quad \int_{J_\sigma} (x - x_0)^2 dP_1 = p_\sigma \left(s_\sigma^2 V + (S_\sigma(\frac{1}{4}) - x_0)^2 \right).$$

4.7. Optimal sets of n -means and n th quantization error. In this subsection, we determine the optimal sets of n -means and the n th quantization errors for all $n \geq 2$ for the mixed distribution P . To determine the distortion error, we will frequently use the equation (3).

Lemma 4.7.1. *Let α be an optimal set of two-means. Then, $\alpha = \{\frac{1}{4}, \frac{5}{6}\}$ with quantization error $V_2 = \frac{43}{1728} = 0.0248843$.*

Proof. Consider the set of two-points β given by $\beta := \{\frac{1}{4}, \frac{5}{6}\}$. Then, the distortion error is

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \frac{1}{2} \int_C (x - \frac{1}{4})^2 dP_1 + \frac{1}{6} \sum_{x \in D} (x - \frac{5}{6})^2 = \frac{43}{1728} = 0.0248843.$$

Since V_2 is the quantization error for two-means, we have $V_2 \leq 0.0248843$. Let $\alpha := \{a_1, a_2\}$ be an optimal set of two-means with $a_1 < a_2$. Since the optimal points are the centroids of their own Voronoi regions, we have $0 < a_1 < a_1 < a_2 \leq 1$. If $a_1 \geq \frac{29}{72} > S_{21}(\frac{1}{2})$, then

$$V_2 \geq \frac{1}{2} \int_{J_1 \cup J_{21}} (x - \frac{29}{72})^2 dP_1 = \frac{1105}{41472} = 0.0266445 > V_2,$$

which leads to a contradiction. We now show that the Voronoi region of a_1 does not contain any point from D . Notice that the Voronoi region of a_1 can not contain all the points from D as by Proposition 1.1, $P(M(a_2|\alpha)) > 0$. First, assume that the Voronoi region of a_1 contains both $\frac{2}{3}$ and $\frac{5}{6}$. Then,

$$a_1 = E(X : X \in C \cup \{\frac{2}{3}, \frac{5}{6}\}) = \frac{\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{2}{3} + \frac{1}{6} \cdot \frac{5}{6}}{\frac{1}{2} + \frac{1}{6} + \frac{1}{6}} = \frac{9}{20} \text{ and } a_2 = 1,$$

which yield $\frac{1}{2}(a_1 + a_2) = \frac{29}{40} < \frac{5}{6}$, which is a contradiction, as we assumed $\{\frac{2}{3}, \frac{5}{6}\} \subset M(a_1|\alpha)$. Next, assume that the Voronoi region of a_1 contains only the point $\frac{2}{3}$ from D . Then,

$$a_1 = E(X : X \in C \cup \{\frac{2}{3}\}) = \frac{\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{2}{3}}{\frac{1}{2} + \frac{1}{6}} = \frac{17}{48} \text{ and } a_2 = \frac{1}{2}(\frac{5}{6} + 1) = \frac{11}{12},$$

which yield $\frac{1}{2}(a_1 + a_2) = \frac{61}{96} < \frac{2}{3}$, which is a contradiction, as the Voronoi region of a_1 contains $\frac{2}{3}$. Thus, we can assume that the Voronoi region of a_1 does not contain any point from D implying that $a_1 \leq \frac{1}{4}$. Notice that if the Voronoi region of a_1 does not contain any point from D and the Voronoi region of a_2 does not contain any point from C , then $a_1 = \frac{1}{4}$ and $a_2 = \frac{5}{6}$. If $a_2 < \frac{21}{32}$, then

$$V_2 \geq \frac{1}{6} \left(\left(\frac{2}{3} - \frac{21}{32} \right)^2 + \left(\frac{5}{6} - \frac{21}{32} \right)^2 + \left(1 - \frac{21}{32} \right)^2 \right) = \frac{1379}{55296} = 0.0249385 > V_2,$$

which gives a contradiction, and so $\frac{21}{32} \leq a_2 \leq \frac{5}{6}$. Suppose that $\frac{21}{32} \leq a_2 \leq \frac{17}{24}$. Since $a_1 \leq \frac{1}{4}$, $E(X_1 : X_1 \in J_1 \cup J_{21}) = \frac{19}{108} < \frac{1}{4}$, and $S_{21}(\frac{1}{2}) < \frac{1}{2}(\frac{19}{108} + \frac{21}{32}) < \frac{1}{2}(\frac{1}{4} + \frac{21}{32}) < S_{2212}(0)$, we have

$$\begin{aligned} V_2 &\geq \frac{1}{2} \left(\int_{J_1 \cup J_{21}} (x - \frac{19}{108})^2 dP_1 + \int_{J_{2212} \cup J_{222}} (x - \frac{21}{32})^2 dP_1 \right) + \frac{1}{6} \left(\left(\frac{5}{6} - \frac{17}{24} \right)^2 + \left(1 - \frac{17}{24} \right)^2 \right) \\ &= \frac{1938409}{71663616} = 0.0270487 > V_2, \end{aligned}$$

which leads to a contradiction. So, we can assume that $\frac{17}{24} \leq a_2 \leq \frac{5}{6}$. Suppose that $\frac{17}{24} \leq a_2 \leq \frac{3}{4}$. Notice that $S_{221}(\frac{1}{2}) < \frac{1}{2}(\frac{1}{4} + \frac{17}{24}) < S_{222}(0)$, and $E(X_1 : X_1 \in J_1 \cup J_{21} \cup J_{221}) = \frac{829}{4212} < \frac{1}{4}$, and so, we have

$$V_2 \geq \frac{1}{2} \left(\int_{J_1 \cup J_{21} \cup J_{221}} (x - \frac{829}{4212})^2 dP_1 + \int_{J_{212}} (x - \frac{1}{4})^2 dP_1 + \int_{J_{222}} (x - \frac{17}{24})^2 dP_1 \right) \\ + \frac{1}{6} \left((\frac{2}{3} - \frac{17}{24})^2 + (\frac{5}{6} - \frac{3}{4})^2 + (1 - \frac{3}{4})^2 \right) = \frac{2242573}{87340032} = 0.0256763 > V_2,$$

which is a contradiction. So, we can assume that $\frac{3}{4} \leq a_2 \leq \frac{5}{6}$. Then, notice that $\frac{1}{2}(a_1 + a_2) < \frac{1}{2}$ implying $a_1 < 1 - a_2 \leq \frac{1}{4}$, but $\frac{1}{2}(\frac{1}{4} + \frac{3}{4}) = \frac{1}{2}$, and thus, P -almost surely the Voronoi region of a_2 does not contain any point from C yielding $a_1 = \frac{1}{4}$, $a_2 = \frac{5}{6}$, and the corresponding quantization error is $V_2 = \frac{43}{1728} = 0.0248843$. \square

Lemma 4.7.2. *Let α be an optimal set of three-means. Then, $\alpha = \{\frac{1}{12}, \frac{31}{60}, \frac{11}{12}\}$ with quantization error $V_3 = \frac{89}{8640} = 0.0103009$.*

Proof. Let us consider the set of three-points $\beta := \{\frac{1}{12}, \frac{31}{60}, \frac{11}{12}\}$. The distortion error due to the set β is given by

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \frac{1}{2} \left(\int_{J_1} (x - \frac{1}{12})^2 dx + \int_{J_2} (x - \frac{31}{60})^2 dx \right) \\ + \frac{1}{6} \left((\frac{2}{3} - \frac{31}{60})^2 + (\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) = \frac{89}{8640} = 0.0103009.$$

Since V_3 is the quantization error for three-means, we have $V_3 \leq 0.0103009$. Let $\alpha := \{a_1, a_2, a_3\}$ be an optimal set of three-means with $a_1 < a_2 < a_3$. Since the optimal points are the centroids of their own Voronoi regions, we have $0 < a_1 < a_2 < a_3 \leq 1$. If $a_3 < \frac{3}{4}$, then

$$V_3 \geq \frac{1}{6} \left((\frac{5}{6} - \frac{3}{4})^2 + (1 - \frac{3}{4})^2 \right) = \frac{5}{432} = 0.0115741 > V_3,$$

which is a contradiction. So, we can assume that $\frac{3}{4} \leq a_3$. We now show that the Voronoi region of a_3 does not contain any point from J_2 . Suppose that the Voronoi region of a_3 contains points from J_2 . Consider the following two cases:

Case 1. $\frac{3}{4} \leq a_3 \leq \frac{5}{6}$.

Then, $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implying $a_2 < 1 - a_3 \leq 1 - \frac{3}{4} = \frac{1}{4}$, and so the distortion error is

$$V_3 \geq \frac{1}{2} \int_{J_2} (x - \frac{1}{4})^2 dP_1 + \frac{1}{6} (1 - \frac{5}{6})^2 = \frac{43}{3456} = 0.0124421 > V_3,$$

which leads to a contradiction.

Case 2. $\frac{5}{6} \leq a_3 < 1$.

Then, $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implying $a_2 < 1 - a_3 \leq 1 - \frac{5}{6} = \frac{1}{6}$, and so the distortion error is

$$V_3 \geq \frac{1}{2} \int_{J_2} (x - \frac{1}{6})^2 dP_1 = \frac{19}{1152} = 0.0164931 > V_3,$$

which is a contradiction.

Thus, by Case 1 and Case 2, we can assume that the Voronoi region of a_3 does not contain any point from J_2 , and so $\frac{5}{6} \leq a_3$. If the Voronoi region of a_2 does not contain any point from D , then we will have $a_1 = a(1)$, $a_2 = a(2)$, $a_3 = \frac{5}{6}$ yielding the distortion error as

$$\frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \int_{J_2} (x - a(2))^2 dP_1 \right) + \frac{1}{6} \sum_{x \in D} (x - \frac{5}{6})^2 = \frac{19}{1728} = 0.0109954 > V_3,$$

which is a contradiction. So, we can assume that the Voronoi region of a_2 contains points from D . If the Voronoi region of a_2 does not contain any point from C , then

$$V_3 \geq \frac{1}{2} \int_C (x - \frac{1}{4})^2 dP_1 = \frac{1}{64} = 0.015625 > V_3,$$

which leads to a contradiction. So, the Voronoi region of a_2 contains points from both C and D . Suppose that the Voronoi region of a_2 contains both $\frac{2}{3}$ and $\frac{5}{6}$ from D . Then, $a_3 = 1$, and $\frac{5}{6} \leq \frac{1}{2}(a_2 + a_3) < 1$ implying $\frac{2}{3} \leq a_2 < 1$. Moreover, as the Voronoi region of a_2 contains points from C , $\frac{1}{2}(a_1 + a_2) < \frac{1}{2}$ implying $a_1 < 1 - a_2 \leq 1 - \frac{2}{3} = \frac{1}{3}$. Notice that $E(X_1 : X_1 \in J_1 \cup J_{21} \cup J_{221}) = \frac{163}{756} < \frac{1}{3}$, and so, we have

$$V_3 \geq \frac{1}{2} \left(\int_{J_1 \cup J_{21} \cup J_{221}} (x - \frac{163}{756})^2 dP_1 + \int_{J_{222}} (x - \frac{1}{3})^2 dP_1 \right) = \frac{17027}{1306368} = 0.0130338 > V_3,$$

which yields a contradiction. Thus, we can assume that the Voronoi region of a_2 contains only the point $\frac{2}{3}$ from D . Then, $a_3 = \frac{1}{2}(\frac{5}{6} + 1) = \frac{11}{12}$, and $\frac{2}{3} \leq \frac{1}{2}(a_2 + a_3) < \frac{5}{6}$ implying $\frac{5}{12} \leq a_2 < \frac{3}{4}$. If the Voronoi region of a_2 contains points from J_1 , then $\frac{1}{2}(a_1 + a_2) < \frac{1}{6}$ implying $a_1 < \frac{1}{3} - a_2 \leq \frac{1}{3} - \frac{5}{12} = -\frac{1}{12}$, which is a contradiction as $0 < a_1$. Thus, the Voronoi region of a_2 does not contain any point from J_1 implying the fact that $a_1 \geq a(1) = \frac{1}{12}$, and $E(X_1 : X_1 \in J_2 \cup \{\frac{2}{3}\}) = \frac{31}{60} \leq a_2 < \frac{3}{4}$. Suppose that $\frac{333}{640} \leq a_2 < \frac{3}{4}$. Then, $\frac{1}{2}(a_1 + a_2) < \frac{1}{2}$ implying $a_1 < 1 - a_2 \leq 1 - \frac{333}{640} = \frac{307}{640} < S_{2222}(0)$. Moreover, $E(X_1 : X_1 \in J_1 \cup J_{21} \cup J_{221} \cup J_{2221}) = \frac{227}{972} < \frac{307}{640}$, and so, writing $A = J_1 \cup J_{21} \cup J_{221} \cup J_{2221}$, we have

$$V_3 \geq \frac{1}{2} \left(\int_A (x - \frac{227}{972})^2 dP_1 + \int_{J_{2222}} (x - \frac{307}{640})^2 dP_1 + \frac{1}{6} \left((\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) \right) = \frac{4106379547}{257989017600},$$

i.e., $V_3 \geq \frac{4106379547}{257989017600} = 0.0159169 > V_3$, which is a contradiction. Thus, we can assume that $\frac{31}{60} \leq a_2 \leq \frac{333}{640}$. If $a_1 \geq \frac{5}{24}$, then,

$$V_3 \geq \frac{1}{2} \int_{J_1} (x - \frac{5}{24})^2 dP_1 + \frac{1}{6} \left((\frac{2}{3} - \frac{333}{640})^2 + (\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) = \frac{78587}{7372800} = 0.010659 > V_3,$$

which gives a contradiction. So, we can assume that $a_1 < \frac{5}{24}$. Suppose that $\frac{1}{6} < a_1 < \frac{5}{24}$. Since, $S_{211}(\frac{1}{2}) < \frac{1}{2}(\frac{5}{24} + \frac{31}{60}) < S_{212}(0)$, we have

$$\begin{aligned} V_3 &\geq \frac{1}{2} \left(\int_{J_1} (x - \frac{1}{6})^2 dP_1 + \int_{J_{211}} (x - \frac{5}{24})^2 dP_1 + \int_{J_{212} \cup J_{22}} (x - \frac{31}{60})^2 dP_1 \right) \\ &+ \frac{1}{6} \left((\frac{2}{3} - \frac{333}{640})^2 + (\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) = \frac{735859}{66355200} = 0.0110897 > V_3, \end{aligned}$$

which leads to a contradiction. So, we can assume that $a_1 \leq \frac{1}{8}$. Suppose that $\frac{1}{8} \leq a_1 \leq \frac{1}{6}$. Then, $S_{2111}(\frac{1}{2}) < \frac{1}{2}(\frac{1}{6} + \frac{31}{60}) < S_{2112}(0)$. Using equation (3), it can be proved that for $\frac{1}{8} \leq a_1 \leq \frac{1}{6}$, the error $\int_{J_1} (x - a_1)^2$ is minimum if $a_1 = \frac{1}{8}$. Thus,

$$\begin{aligned} V_3 &\geq \frac{1}{2} \left(\int_{J_1} (x - \frac{1}{8})^2 dP_1 + \int_{J_{2111}} (x - \frac{1}{6})^2 dP_1 + \int_{J_{2112} \cup J_{212} \cup J_{22}} (x - \frac{31}{60})^2 dP_1 \right) \\ &+ \frac{1}{6} \left((\frac{2}{3} - \frac{333}{640})^2 + (\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) = \frac{138551}{13271040} = 0.0104401 > V_3, \end{aligned}$$

which leads to a contradiction. So, we can assume that $a_1 \leq \frac{1}{8}$. Then, notice that $\frac{1}{2}(\frac{1}{8} + \frac{31}{60}) < \frac{1}{3} = S_2(0)$, i.e., the Voronoi region of a_1 does not contain any point from J_2 , implying $a_1 = a(1) = \frac{1}{12}$, $a_2 = \frac{31}{60}$, and $a_3 = \frac{11}{12}$, and the corresponding quantization error is given by $V_3 = \frac{89}{8640} = 0.0103009$. Thus, the lemma is yielded. \square

Lemma 4.7.3. *Let α be an optimal set of four-means. Then, $\alpha = \{\frac{1}{12}, \frac{5}{12}, \frac{3}{4}, 1\}$, or $\alpha = \{\frac{1}{12}, \frac{5}{12}, \frac{2}{3}, \frac{11}{12}\}$, and the quantization error is $V_4 = \frac{7}{1728} = 0.00405093$.*

Proof. Let us consider the set of four points $\beta := \{\frac{1}{12}, \frac{5}{12}, \frac{3}{4}, 1\}$. Then, the distortion error due to the set β is

$$\begin{aligned} & \int \min_{b \in \beta} \|x - b\|^2 dP \\ &= \frac{1}{2} \left(\int_{J_1} (x - \frac{1}{12})^2 dP_1 + \int_{J_2} (x - \frac{5}{12})^2 dP_1 \right) + \frac{1}{6} \left((\frac{2}{3} - \frac{3}{4})^2 + (\frac{5}{6} - \frac{3}{4})^2 \right) = \frac{7}{1728}. \end{aligned}$$

Since V_4 is the quantization error for four-means, we have $V_4 \leq \frac{7}{1728} = 0.00405093$. Let $\alpha := \{a_1 < a_2 < a_3 < a_4\}$ be an optimal set of four-means. Since the optimal points are the centroids of their own Voronoi regions, we have $0 < a_1 < a_2 < a_3 < a_4 \leq 1$. If $a_1 \geq \frac{19}{96}$, then

$$V_4 \geq \frac{1}{2} \int_{J_1} (x - \frac{19}{96})^2 dP_1 = \frac{17}{4096} = 0.00415039 > V_4,$$

which is a contradiction. So, we can assume that $a_1 < \frac{19}{96}$. Suppose that the Voronoi region of a_2 contains points from D . Then, it contains only the point $\frac{2}{3}$ from D , as it must be $a_3 = \frac{5}{6}$, and $a_4 = 1$. Moreover, $\frac{2}{3} \leq \frac{1}{2}(a_2 + a_3) < \frac{5}{6}$ implying $\frac{1}{2} \leq a_2 < \frac{5}{6}$. Then, $S_{21121}(\frac{1}{2}) < \frac{1}{2}(\frac{19}{96} + \frac{1}{2}) < S_{21122}(0)$ yielding

$$V_4 \geq \frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \int_{J_{2111} \cup J_{21121}} (x - \frac{19}{96})^2 dP_1 + \int_{J_{21122} \cup J_{212} \cup J_{22}} (x - \frac{1}{2})^2 dP_1 \right) = \frac{153563}{23887872},$$

i.e., $V_4 \geq \frac{153563}{23887872} = 0.00642849 > V_4$, which gives a contradiction. Therefore, we can assume that the Voronoi region of a_2 does not contain any point from D . If the Voronoi region of a_3 does not contain any point from D , then

$$V_4 \geq \frac{1}{6} \left((\frac{2}{3} - \frac{5}{6})^2 + (1 - \frac{5}{6})^2 \right) = \frac{1}{108} = 0.00925926 > V_4,$$

which leads to a contradiction. So, the Voronoi region of a_3 contains points from D . Suppose that the Voronoi region of a_3 contains points from C as well. Then, two cases can arise.

Case 1. $\{\frac{2}{3}\} \subset M(a_3|\alpha)$ and $\{\frac{5}{6}, 1\} \subset M(a_4|\alpha)$.

Then, $a_4 = \frac{1}{2}(\frac{5}{6} + 1) = \frac{11}{12}$, and $\frac{2}{3} \leq \frac{1}{2}(a_3 + a_4) < \frac{5}{6}$ implying $\frac{5}{12} \leq a_3 < \frac{3}{4}$. Assume that $a_3 < \frac{9}{16}$. Then,

$$V_4 \geq \frac{1}{6} \left((\frac{2}{3} - \frac{9}{16})^2 + (\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) = \frac{19}{4608} = 0.00412326 > V_4,$$

which gives a contradiction. So, we can assume that $\frac{9}{16} \leq a_3 < \frac{3}{4}$. Suppose that $\frac{9}{16} \leq a_3 < \frac{7}{12}$. Then, if $a_2 \leq \frac{1}{3}$, as $S_{22111}(\frac{1}{2}) < \frac{1}{2}(\frac{1}{3} + \frac{9}{16}) < S_{22112}(0)$, we have

$$\begin{aligned} V_4 &\geq \frac{1}{2} \left(\int_{J_{21} \cup J_{22111}} (x - \frac{1}{3})^2 + \int_{J_{22112} \cup J_{2212} \cup J_{222}} (x - \frac{9}{16})^2 \right) \\ &+ \frac{1}{6} \left((\frac{2}{3} - \frac{7}{12})^2 + (\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) = \frac{55735}{11943936} = 0.00466638 > V_4, \end{aligned}$$

which leads to a contradiction. If $\frac{1}{3} < a_2$, then the Voronoi region of a_2 does not contain any point from J_1 , and $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implies that $a_2 < 1 - a_3 \leq 1 - \frac{9}{16} = \frac{7}{16} < S_{22}(0)$, and so, we have

$$\begin{aligned} V_4 &\geq \frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \int_{J_{22}} (x - \frac{7}{16})^2 dP_1 \right) \\ &+ \frac{1}{6} \left((\frac{2}{3} - \frac{7}{12})^2 + (\frac{5}{6} - \frac{11}{12})^2 + (1 - \frac{11}{12})^2 \right) = \frac{251}{55296} = 0.00453921 > V_4, \end{aligned}$$

which is a contradiction. So, we can assume that $\frac{7}{12} \leq a_3 < \frac{3}{4}$. Suppose that $\frac{7}{12} \leq a_3 \leq \frac{29}{48}$. Then, $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implying $a_2 < 1 - a_3 \leq \frac{5}{12}$. First, assume that $\frac{1}{3} \leq a_2 < \frac{5}{12}$. Then, the

Voronoi region of a_2 does not contain any point from J_1 . Moreover, using equation (3), we see that for $\frac{1}{3} \leq a_2 < \frac{5}{12}$, the error $\int_{J_2} (x - a_2)^2 dP_1$ is minimum if $a_2 = \frac{5}{12}$, and so,

$$\begin{aligned} V_4 &\geq \frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \int_{J_2} (x - \frac{5}{12})^2 dP_1 \right) + \frac{1}{6} \left(\left(\frac{2}{3} - \frac{29}{48} \right)^2 + \left(\frac{5}{6} - \frac{11}{12} \right)^2 + \left(1 - \frac{11}{12} \right)^2 \right) \\ &= \frac{65}{13824} = 0.00470197 > V_4, \end{aligned}$$

which gives a contradiction. Next, assume that $a_2 < \frac{1}{3}$. Then, $S_{221212}(0) < \frac{1}{2}(\frac{1}{3} + \frac{7}{12}) < S_{221212}(\frac{1}{2})$ implying

$$\begin{aligned} V_4 &\geq \frac{1}{2} \left(\int_{J_{21} \cup J_{2211} \cup J_{221211}} (x - \frac{1}{3})^2 dP_1 + \int_{J_{22122} \cup J_{222}} (x - \frac{7}{12})^2 dP_1 \right) \\ &\quad + \frac{1}{6} \left(\left(\frac{2}{3} - \frac{29}{48} \right)^2 + \left(\frac{5}{6} - \frac{11}{12} \right)^2 + \left(1 - \frac{11}{12} \right)^2 \right) = \frac{3197515}{725594112} = 0.00440675 > V_4, \end{aligned}$$

which leads to a contradiction. So, we can assume that $\frac{29}{48} \leq a_3 < \frac{3}{4}$. Then, $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implies $a_2 < 1 - a_3 \leq \frac{19}{48}$. First, assume that $\frac{1}{3} \leq a_2 < \frac{19}{48}$. Then, the Voronoi region of a_2 does not contain any point from J_1 . Moreover, using equation (3), we see that for $\frac{1}{3} \leq a_2 < \frac{19}{48}$, the error $\int_{J_2} (x - a_2)^2 dP_1$ is minimum if $a_2 = \frac{19}{48}$, and so,

$$\begin{aligned} V_4 &\geq \frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \frac{1}{2} \int_{J_2} (x - \frac{19}{48})^2 dP_1 \right) + \frac{1}{6} \left(\left(\frac{5}{6} - \frac{11}{12} \right)^2 + \left(1 - \frac{11}{12} \right)^2 \right) \\ &= \frac{115}{27648} = 0.00415943 > V_4, \end{aligned}$$

which gives a contradiction. Next, assume that $a_2 \leq \frac{1}{3}$, then $S_{221}(\frac{1}{2}) < \frac{1}{2}(\frac{1}{3} + \frac{29}{48}) < S_{222}(0)$ implying

$$V_4 \geq \frac{1}{2} \left(\int_{J_{21} \cup J_{221}} (x - \frac{1}{3})^2 dP_1 + \int_{J_{222}} (x - \frac{29}{48})^2 dP_1 \right) + \frac{1}{6} \left(\left(\frac{5}{6} - \frac{11}{12} \right)^2 + \left(1 - \frac{11}{12} \right)^2 \right) = \frac{1385}{331776},$$

i.e., $V_4 \geq \frac{1385}{331776} = 0.0041745 > V_4$, which yields a contradiction.

Case 2. $\{\frac{2}{3}, \frac{5}{6}\} \subset M(a_3|\alpha)$ and $a_4 = 1$.

Then, $\frac{5}{6} \leq \frac{1}{2}(a_3 + 1)$ implying $a_3 \geq \frac{5}{3} - 1 = \frac{2}{3}$. Since by the assumption, the Voronoi region of a_3 contains points from C , we have $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implying $a_2 < 1 - a_3 \leq 1 - \frac{2}{3} = \frac{1}{3}$. Then,

$$V_4 \geq \frac{1}{2} \int_{J_2} (x - \frac{1}{3})^2 dP_1 + \frac{1}{6} \left(\left(\frac{2}{3} - \frac{3}{4} \right)^2 + \left(\frac{5}{6} - \frac{3}{4} \right)^2 \right) = \frac{17}{3456} = 0.00491898 > V_4,$$

which is a contradiction.

Thus, by Case 1 and Case 2, we can assume that the Voronoi region of a_3 does not contain any point from C . Again, we have proved that the Voronoi region of a_2 does not contain any point from D . Hence, $(a_1 = a(1), a_2 = a(2), a_3 = \frac{3}{4} \text{ and } a_4 = 1)$, or $(a_1 = a(1), a_2 = a(2), a_3 = \frac{2}{3} \text{ and } a_4 = \frac{11}{12})$, and the corresponding quantization error is $V_4 = \frac{7}{1728} = 0.00405093$, which is the lemma. \square

Lemma 4.7.4. *Let α be an optimal set of five-means. Then, $\alpha = \alpha_2(P_1) \cup D$, and the corresponding quantization error is $V_5 = \frac{1}{576} = \frac{1}{2}V_2(P_1)$.*

Proof. Consider the set of five points $\beta := \{\frac{1}{12}, \frac{5}{12}, \frac{2}{3}, \frac{5}{6}, 1\}$. The distortion error due to the set β is given by

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \frac{1}{2} \int_{J_1} (x - \frac{1}{12})^2 dP_1 + \frac{1}{2} \int_{J_2} (x - \frac{5}{12})^2 dP_1 = \frac{1}{576} = 0.00173611.$$

Since V_5 is the quantization error for five-means, we have $V_5 \leq 0.00173611$. Let $\alpha := \{a_1 < a_2 < a_3 < a_4 < a_5\}$ be an optimal set of five-means. Since the optimal points are the centroids of

their own Voronoi regions, we have $0 < a_1 < a_2 < a_3 < a_4 < a_5 \leq 1$. Suppose that $\frac{1}{6} \leq a_1$. Then,

$$V_5 \geq \frac{1}{2} \int_{J_1} (x - \frac{1}{6})^2 dP_1 = \frac{1}{384} = 0.00260417 > V_5,$$

which is a contradiction. So, we can assume that $a_1 < \frac{1}{6}$. If the Voronoi region of a_1 contains points from J_2 , we must have $\frac{1}{2}(a_1 + a_2) > \frac{1}{3}$ implying $a_2 > \frac{2}{3} - a_1 > \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$, and then the distortion error is

$$V_5 \geq \frac{1}{2} \left(\int_{J_1} (x - \frac{1}{12})^2 dP_1 + \int_{J_2} (x - \frac{1}{2})^2 dP_1 \right) = \frac{1}{288} = 0.00347222 > V_5,$$

which leads to a contradiction. So, the Voronoi region of a_1 does not contain any point from J_2 implying $a_1 \leq \frac{1}{12}$. Notice that the Voronoi region of a_2 can not contain any point from D , as α is an optimal set of five-means and D contains only three points. Thus, we have $a_2 \leq a(2) = \frac{5}{12}$. If the Voronoi region of a_3 does not contain any point from D , then

$$V_5 \geq \frac{1}{6} \left(\left(\frac{2}{3} - \frac{5}{6} \right)^2 + \left(1 - \frac{5}{6} \right)^2 \right) = \frac{1}{108} = 0.00925926 > V_5,$$

which is a contradiction. So, we can assume that the Voronoi region of a_3 contains a point from D . In that case, we must have $a_4 = \frac{5}{6}$ and $a_5 = 1$. If the Voronoi region of a_3 does not contain any point from C , then $a_3 = \frac{2}{3}$. Suppose that the Voronoi region of a_3 contains points from C . Then, $\frac{2}{3} \leq \frac{1}{2}(a_3 + a_4)$ implying $a_3 \geq \frac{4}{3} - a_4 = \frac{4}{3} - \frac{5}{6} = \frac{1}{2}$, i.e., $\frac{1}{2} \leq a_3 \leq \frac{2}{3}$. The following three cases can arise:

Case A. $\frac{1}{2} \leq a_3 \leq \frac{7}{12}$.

If $a_2 < \frac{7}{24}$, then $S_{21}(\frac{1}{2}) < \frac{1}{2}(\frac{7}{24} + \frac{1}{2}) < S_{22}(0)$ yielding

$$V_5 \geq \frac{1}{2} \left(\int_{J_{21}} (x - \frac{7}{24})^2 dP_1 + \int_{J_{22}} (x - \frac{1}{2})^2 dP_1 \right) + \frac{1}{6} \left(\frac{2}{3} - \frac{7}{12} \right)^2 = \frac{1}{512} = 0.00195313 > V_5,$$

which leads to a contradiction. Assume that $\frac{7}{24} \leq a_2 \leq \frac{1}{3}$. Then, $\frac{1}{2}(a_1 + a_2) < \frac{1}{6}$ implying $a_1 < \frac{1}{3} - a_2 \leq \frac{1}{3} - \frac{7}{24} = \frac{1}{24}$, and so,

$$V_5 \geq \frac{1}{2} \left(\int_{J_{12}} (x - \frac{1}{24})^2 dP_1 + \int_{J_{21}} (x - \frac{1}{3})^2 dP_1 + \int_{J_{22}} (x - \frac{1}{2})^2 dP_1 \right) + \frac{1}{6} \left(\frac{2}{3} - \frac{7}{12} \right)^2 = \frac{37}{13824},$$

i.e., $V_5 \geq \frac{37}{13824} = 0.0026765 > V_5$, which is a contradiction. Next, assume that $\frac{1}{3} < a_2$, and then, the Voronoi region of a_2 does not contain any point from J_1 . Recall that $a_2 \leq \frac{5}{12}$. Thus, we have

$$V_5 \geq \frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \int_{J_{21}} (x - a(21))^2 dP_1 + \int_{J_{22}} (x - \frac{1}{2})^2 dP_1 \right) + \frac{1}{6} \left(\frac{2}{3} - \frac{7}{12} \right)^2 = \frac{23}{10368},$$

i.e., $V_5 \geq \frac{23}{10368} = 0.00221836 > V_5$, which gives a contradiction.

Case B. $\frac{7}{12} \leq a_3 \leq \frac{5}{8}$.

As Case A, we can show that if $a_2 < \frac{7}{24}$ a contradiction arises. Assume that $\frac{7}{24} \leq a_1 \leq \frac{1}{3}$, then $S_{221212}(0) < \frac{1}{2}(\frac{1}{3} + \frac{7}{12}) < S_{221212}(\frac{1}{2})$, and so,

$$\begin{aligned} V_5 &\geq \frac{1}{2} \left(\int_{J_{11}} (x - a(11))^2 dP_1 + \int_{J_{21} \cup J_{2211} \cup J_{221211}} (x - \frac{1}{3})^2 dP_1 + \int_{J_{22122} \cup J_{222}} (x - \frac{7}{12})^2 dP_1 \right) \\ &+ \frac{1}{6} \left(\frac{2}{3} - \frac{5}{8} \right)^2 = \frac{1290451}{725594112} = 0.00177848 > V_5, \end{aligned}$$

which give a contradiction. Next, assume that $\frac{1}{3} < a_2$, and then the Voronoi region of a_2 does not contain any point from J_1 . Again, $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implies that $a_2 < 1 - a_3 \leq 1 - \frac{7}{12} = \frac{5}{12}$. Moreover, for $\frac{1}{3} < a_2 \leq \frac{5}{12}$, the error $\int_{J_2} (x - a_2)^2 dP_1$ is minimum if $a_2 = \frac{5}{12}$. Thus,

$$V_3 \geq \frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \int_{J_2} (x - \frac{5}{12})^2 dP_1 \right) + \frac{1}{6} \left(\frac{2}{3} - \frac{5}{8} \right)^2 = \frac{7}{3456} = 0.00202546 > V_5,$$

which leads to a contradiction.

By Case A and Case B, we can assume that $\frac{5}{8} \leq a_3 \leq \frac{2}{3}$. We now show that the Voronoi region of a_3 does not contain any point from C . On the contrary, assume that $\frac{1}{2}(a_2 + a_3) < \frac{1}{2}$ implying $a_2 < 1 - a_3 \leq 1 - \frac{5}{8} = \frac{3}{8}$. Then, if $a_2 < \frac{1}{3}$, as $S_{221}(\frac{1}{2}) < \frac{1}{2}(\frac{1}{3} + \frac{5}{8}) < S_{222}(0)$, we have

$$V_5 \geq \frac{1}{2} \left(\int_{J_{21} \cup J_{221}} (x - \frac{1}{3})^2 dP_1 + \int_{J_{222}} (x - \frac{5}{8})^2 dP_1 \right) = \frac{181}{82944} = 0.0021822 > V_5$$

which gives a contradiction. Assume that $\frac{1}{3} < a_2$. Then, the Voronoi region of a_2 does not contain any point from J_1 . Using equation (3), we can show that for $\frac{1}{3} \leq a_2 \leq \frac{3}{8}$, the error $\int_{J_2} (x - a_2)^2 dP_1$ is minimum if $a_2 = \frac{3}{8}$, and so

$$V_5 \geq \frac{1}{2} \left(\int_{J_1} (x - a(1))^2 dP_1 + \int_{J_2} (x - \frac{3}{8})^2 dP_1 \right) = \frac{5}{2304} = 0.00217014,$$

which is a contradiction. So, we can assume that the Voronoi region of a_3 does not contain any point from C yielding $a_1 = a(1)$, $a_2 = a(2)$, $a_3 = \frac{2}{3}$, $a_4 = \frac{5}{6}$, and $a_5 = 1$, and so, by Proposition 4.4, we have $\alpha = \alpha_2(P_1) \cup D$, and the corresponding quantization error is $V_5 = \frac{1}{576} = \frac{1}{2}V_2(P_1)$. Thus, the proof of the lemma is complete. \square

Theorem 4.7.5. *Let $n \in \mathbb{N}$ and $n \geq 5$, and let α_n be an optimal set of n -means for P and $\alpha_n(P_1)$ be the optimal set of n -means for P_1 . Then,*

$$\alpha_n(P) = \alpha_{n-3}(P_1) \cup D, \text{ and } V_n(P) = \frac{1}{2}V_{n-3}(P_1).$$

Proof. If $n = 5$, by Lemma 4.7.4, we see that the theorem is true for $n = 5$. Proceeding in the similar way, as Lemma 4.7.4, we can show that the theorem is true for $n = 6$ and $n = 7$. We now show that the theorem is true for all $n \geq 8$. Consider the set of eight points $\beta := \{a(11), a(12), a(21), a(221), a(222), \frac{2}{3}, \frac{5}{6}, 1\}$. The distortion error due to set β is given by

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \frac{1}{2}V_5(P_1) = \frac{7}{46656} = 0.000150034.$$

Since V_n is the n th quantization error for n -means for $n \geq 8$, we have $V_n \leq V_8 \leq 0.000150034$. Let $\alpha_n := \{a_1 < a_2 < \dots < a_n\}$ be an optimal set of n -means for $n \geq 8$, where $0 < a_1 < \dots < a_n \leq 1$. To prove the first part of the theorem, it is enough to show that $M(a_{n-2}|\alpha_n)$ does not contain any point from C , and $M(a_{n-3}|\alpha_n)$ does not contain any point from D . If $M(a_{n-2}|\alpha_n)$ does not contain any point from D , then

$$V_n \geq \frac{1}{6} \left(\left(\frac{2}{3} - \frac{3}{4} \right)^2 + \left(\frac{5}{6} - \frac{3}{4} \right)^2 \right) = \frac{1}{432} = 0.00231481 > V_n,$$

which leads to a contradiction. So, $M(a_{n-2}|\alpha_n)$ contains a point, in fact the point $\frac{2}{3}$, from D . If $M(a_{n-2}|\alpha_n)$ does not contain points from C , then $a_{n-2} = \frac{2}{3}$. Suppose that $M(a_{n-2}|\alpha_n)$ contains points from C . Then, $\frac{2}{3} \leq \frac{1}{2}(a_{n-2} + a_{n-1})$ implies $a_{n-2} \geq \frac{4}{3} - a_{n-1} = \frac{4}{3} - \frac{5}{6} = \frac{1}{2}$. The following three cases can arise:

Case 1. $\frac{1}{2} \leq a_{n-2} \leq \frac{7}{12}$.

Then, $V_n \geq \frac{1}{6} \left(\frac{2}{3} - \frac{7}{12} \right)^2 = \frac{1}{864} = 0.00115741 > V_n$, which is a contradiction.

Case 2. $\frac{7}{12} \leq a_{n-2}$.

Then, $\frac{1}{2}(a_{n-3} + a_{n-2}) < \frac{1}{2}$ implying $a_{n-3} < 1 - a_{n-2} \leq 1 - \frac{7}{12} = \frac{5}{12}$, and so

$$V_n \geq \frac{1}{2} \int_{J_{22}} \left(x - \frac{5}{12} \right)^2 dP_1 = \frac{1}{2304} = 0.000434028 > V_n,$$

which leads to a contradiction.

By Case 1 and Case 2, we can assume that $M(a_{n-2}|\alpha_n)$ does not contain any point from C . If $M(a_{n-3}|\alpha)$ contains any point from D , say $\frac{2}{3}$, then we will have

$$M(a_{n-2}|\alpha) \cup M(a_{n-1}|\alpha) \cup M(a_n|\alpha) = \{\frac{5}{6}, 1\},$$

which by Proposition 1.1 implies that either $(a_{n-2} = a_{n-1} = \frac{5}{6}$ and $a_n = 1)$, or $(a_{n-2} = \frac{5}{6}$ and $a_{n-1} = a_n = 1)$, which contradicts the fact that $0 < a_1 < \dots < a_{n-2} < a_{n-1} < a_n \leq 1$. Thus, $M(a_{n-3}|\alpha)$ does not contain any point from D . Hence, $\alpha_n(P) = \alpha_{n-3}(P_1) \cup D$, and so,

$$V_n(P) = \frac{1}{2} \int_C \min_{a \in \alpha_{n-3}(P_1)} (x - a)^2 dP_1 + \frac{1}{6} \sum_{x \in D} \min_{a \in D} (x - a)^2 = \frac{1}{2} \int_C \min_{a \in \alpha_{n-3}(P_1)} (x - a)^2 dP_1$$

implying $V_n(P) = \frac{1}{2} V_{n-3}(P_1)$. Thus, the proof of the theorem is complete. \square

Remark 4.7.6. Let β be the Hausdorff dimension of the Cantor set generated by the similarity mappings S_1 and S_2 . Then, $\beta = \frac{\log 2}{\log 3}$. By [GL2, Theorem 6.6], it is known that the quantization dimension of P_1 exists and equals β , i.e., $D(P_1) = \beta$. Since

$$D(P) = \lim_{n \rightarrow \infty} \frac{2 \log n}{-\log 2 - \log V_{n-m}(P_1)} = \lim_{n \rightarrow \infty} \frac{2 \log(n-m)}{-\log V_{n-m}(P_1)} = D(P_1) = \beta,$$

we can say that the quantization dimension of the mixed distribution exists and equals the quantization dimension of the Cantor distribution P_1 , i.e., $D(P) = D(P_1) = \beta$. Again, by [GL2, Theorem 6.3], it is known that the quantization coefficient for P_1 does not exist. By Theorem 4.7.5, we have $\liminf_{n \rightarrow \infty} n^{\frac{2}{\beta}} V_n(P) = \frac{1}{2} \liminf_{n \rightarrow \infty} n^{\frac{2}{\beta}} V_{n-3}(P_1) = \frac{1}{2} \liminf_{n \rightarrow \infty} (n-3)^{\frac{2}{\beta}} V_{n-3}(P_1)$, and similarly, $\limsup_{n \rightarrow \infty} n^{\frac{2}{\beta}} V_n(P) = \frac{1}{2} \limsup_{n \rightarrow \infty} (n-3)^{\frac{2}{\beta}} V_{n-3}(P_1)$. Hence, the quantization coefficient for the mixed distribution P does not exist.

5. SOME REMARKS

Theorem 2.6.5 and Theorem 4.7.5 motivate us to give the following remarks.

Remark 5.1. Let $0 < p < 1$ be fixed. Let P be the mixed distribution given by $P = pP_1 + (1-p)P_2$ with the support of $P_1 = C$ and the support of $P_2 = D$, such that P_1 is continuous on C and P_2 is discrete on D . Let $\text{card}(D) = m$ for some positive integer m . Further assume that C and D are *strongly separated*: there exists a $\delta > 0$ such that $d(C, D) := \inf\{d(x, y) : x \in C \text{ and } y \in D\} > \delta$. Then, there exists a positive integer N such that for all $n \geq N$, we have $\alpha_n(P) = \alpha_{n-m}(P_1) \cup D$, and so

$$V_n(P) = \int \min_{a \in \alpha_n(P)} (x - a)^2 dP = p \int \min_{a \in \alpha_{n-m}(P_1)} (x - a)^2 dP_1 + \sum_{x \in D} \min_{a \in D} (x - a)^2 h(x),$$

implying

$$V_n(P) = p \int \min_{a \in \alpha_{n-m}(P_1)} (x - a)^2 dP_1 = p V_{n-m}(P_1).$$

Thus, we have

$$D(P) = \lim_{n \rightarrow \infty} \frac{2 \log n}{-\log p - \log V_{n-m}(P_1)} = \lim_{n \rightarrow \infty} \frac{2 \log(n-m)}{-\log V_{n-m}(P_1)} = D(P_1).$$

Remark 5.2. Let D be a finite discrete subset of $C := [0, 1]$. If P_1 is continuous on C , singular or nonsingular, and P_2 is discrete on D , then for the mixed distribution $P := pP_1 + (1-p)P_2$, where $0 < p < 1$, the optimal sets of n -means and the n th quantization errors for all $n \geq 2$ and for all D are not known yet. Some special cases to be investigated are as follows: Take $p = \frac{1}{2}$, P_1 as a uniform distribution on C , and $D = \{\frac{2}{3}, \frac{5}{6}, 1\}$. The optimal sets of n -means and the n th quantization errors for such a mixed distribution for all $n \geq 2$ are not known yet. Such a problem can also be investigated by taking P_1 as a Cantor distribution, and P_2 discrete on

D , for example, one can take P_1 the classical Cantor distribution, as considered in [GL2], and $D = \{\frac{2}{3}, \frac{5}{6}, 1\}$. Notice that p , P_1 and D can be chosen in many different ways.

6. QUANTIZATION WHERE P_1 AND P_2 ARE CANTOR DISTRIBUTIONS

Let P_1 be the Cantor distribution given by $P_1 = \frac{1}{2}P_1 \circ S_1^{-1} + \frac{1}{2}P_2 \circ S_2^{-1}$, where $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{9}$ for all $x \in \mathbb{R}$. Let P_2 be the Cantor distribution given by $P_2 = \frac{1}{2}P_2 \circ T_1^{-1} + \frac{1}{2}P_2 \circ T_2^{-1}$, where $T_1(x) = \frac{1}{4}x + \frac{1}{2}$ and $T_2(x) = \frac{1}{4}x + \frac{3}{4}$ for all $x \in \mathbb{R}$. Let C be the Cantor set generated by S_1 and S_2 , and D be the Cantor set generated by T_1 and T_2 . Let P be the mixed distribution generated by P_1 and P_2 such that $P = \frac{1}{2}P_1 + \frac{1}{2}P_2$. Let $\{1, 2\}^*$ be the set of all words over the alphabet $\{1, 2\}$ including the empty word \emptyset as defined in Section 4. Write $J := [0, \frac{1}{3}]$ and $K := [\frac{2}{3}, 1]$. Then, we have $C = \bigcap_{k \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2\}^k} J_\sigma$ and $D = \bigcap_{k \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2\}^k} K_\sigma$, where for $\sigma \in \{1, 2\}^*$, $J_\sigma = S_\sigma([0, \frac{1}{3}])$ and $K_\sigma = T_\sigma([\frac{2}{3}, 1])$. Thus, C is the support of P_1 , and D is the support of P_2 implying the fact that $C \cup D$ is the support of the mixed distribution P . As before, if nothing is mentioned within a parenthesis, by α_n and V_n , we mean an optimal set of n -means and the corresponding quantization error for the mixed distribution P .

The following two lemmas are similar to Lemma 4.2.

Lemma 6.1. *Let $E(P_1)$ and $V(P_1)$ denote the expected value and the variance of a P_1 -distributed random variable. Then, $E(P_1) = \frac{1}{6}$ and $V(P_1) = \frac{1}{72}$. Moreover, for any $x_0 \in \mathbb{R}$, $\int (x - x_0)^2 dP_1 = V(P_1) + (x_0 - \frac{1}{6})^2$.*

Lemma 6.2. *Let $E(P_2)$ and $V(P_2)$ denote the expected value and the variance of a P_2 -distributed random variable. Then, $E(P_2) = \frac{5}{6}$ and $V(P_2) = \frac{1}{60}$. Moreover, for any $x_0 \in \mathbb{R}$, $\int (x - x_0)^2 dP_2(x) = V(P_2) + (x_0 - \frac{5}{6})^2$.*

We now prove the following lemma.

Lemma 6.3. *Let $E(P)$ and $V(P)$ denote the expected value and the variance of a P -distributed random variable, where P is the mixed distribution given by $P = \frac{1}{2}P_1 + \frac{1}{2}P_2$. Then, $E(P) = \frac{1}{2}$ and $V(P) = \frac{91}{720}$. Moreover, for any $x_0 \in \mathbb{R}$, $\int (x - x_0)^2 dP(x) = V(P) + (x_0 - \frac{1}{2})^2$.*

Proof. Let X be a P -distributed random variable. Then,

$$E(X) = \int x dP(x) = \frac{1}{2} \int x dP_1 + \frac{1}{2} \int x dP_2(x) = \frac{1}{2} \left(\frac{1}{6} + \frac{5}{6} \right) = \frac{1}{2}, \text{ and}$$

$$E(X^2) = \int x^2 dP(x) = \frac{1}{2} \int x^2 dP_1 + \frac{1}{2} \int x^2 dP_2(x) = \frac{1}{2} \left(\frac{1}{24} + \frac{32}{45} \right) = \frac{271}{720},$$

and so, $V(P) = E(X^2) - (E(X))^2 = \frac{91}{720}$. Then, by the standard theory of probability, for any $x_0 \in \mathbb{R}$, $\int (x - x_0)^2 dP(x) = V(P) + (x_0 - \frac{1}{2})^2$. Thus, the proof of the lemma is complete. \square

Remark 6.4. From Lemma 6.3, it follows that the optimal set of one-mean for the mixed distribution P is $\frac{1}{2}$ and the corresponding quantization error is $V(P) = \frac{91}{720}$. Again, notice that for any $x_0 \in \mathbb{R}$, we have

$$\int (x - x_0)^2 dP(x) = \frac{1}{2} \left(V(P_1) + V(P_2) + (x_0 - \frac{1}{6})^2 + (x_0 - \frac{5}{6})^2 \right).$$

Definition 6.5. For $n \in \mathbb{N}$ with $n \geq 2$, let $\ell(n)$ be the unique natural number with $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. For $\sigma \in \{1, 2\}^*$, let $a(\sigma)$ and $b(\sigma)$, respectively, denote the midpoints of the basic intervals J_σ and K_σ . Let $I \subset \{1, 2\}^{\ell(n)}$ with $\text{card}(I) = n - 2^{\ell(n)}$. Define $\beta_n(P_1, I)$ and $\beta_n(P_2, I)$ as follows:

$$\beta_n(P_1, I) = \{a(\sigma) : \sigma \in \{1, 2\}^{\ell(n)} \setminus I\} \cup \{a(\sigma 1) : \sigma \in I\} \cup \{a(\sigma 2) : \sigma \in I\}, \text{ and}$$

$$\beta_n(P_2, I) = \{b(\sigma) : \sigma \in \{1, 2\}^{\ell(n)} \setminus I\} \cup \{b(\sigma 1) : \sigma \in I\} \cup \{b(\sigma 2) : \sigma \in I\}.$$

The following proposition follows due to [GL2, Definition 3.5 and Proposition 3.7].

Proposition 6.6. *Let $\beta_n(P_1, I)$ and $\beta_n(P_2, I)$ be the sets for $n \geq 2$ given by Definition 6.5. Then, $\beta_n(P_1, I)$ and $\beta_n(P_2, I)$ form optimal sets of n -means for P_1 and P_2 , respectively, and the corresponding quantization errors are given by*

$$V_n(P_1) = \int \min_{a \in \beta_n(P_1, I)} \|x - a\|^2 dP_1 = \frac{1}{18^{\ell(n)}} \cdot \frac{1}{72} \left(2^{\ell(n)+1} - n + \frac{1}{9} (n - 2^{\ell(n)}) \right), \text{ and}$$

$$V_n(P_2) = \int \min_{a \in \beta_n(P_2, I)} \|x - a\|^2 dP_2 = \frac{1}{32^{\ell(n)}} \cdot \frac{1}{60} \left(2^{\ell(n)+1} - n + \frac{1}{16} (n - 2^{\ell(n)}) \right).$$

Proposition 6.7. *For $n \geq 2$, let α_n be an optimal set of n -means for P . Then, $\alpha_n \cap [0, \frac{1}{3}] \neq \emptyset$ and $\alpha_n \cap (\frac{2}{3}, 1] \neq \emptyset$.*

Proof. Consider the set of two-points $\beta_2 := \{\frac{1}{6}, \frac{5}{6}\}$. Then,

$$\int \min_{a \in \beta_2} \|x - a\|^2 dP = \frac{1}{2} \left(\int (x - \frac{1}{6})^2 dP_1 + \int (x - \frac{5}{6})^2 dP_2 \right) = \frac{11}{720} = 0.0152778.$$

Since V_n is the quantization error for n -means for $n \geq 2$, we have $V_n \leq V_2 \leq 0.0152778$. Let $\alpha_n = \{a_1, a_2, a_3, \dots, a_n\}$ be an optimal set of n -means such that $a_1 < a_2 < a_3 < \dots < a_n$. Since the optimal points are centroids of their own Voronoi regions, we have $0 < a_1 < \dots < a_n < 1$. Assume that $\frac{1}{3} \leq a_1$. Then,

$$V_n \geq \int_{[0, \frac{1}{3}]} (x - \frac{1}{3})^2 dP = \frac{1}{2} \int_{[0, \frac{1}{3}]} (x - \frac{1}{3})^2 dP_1 = \frac{1}{48} = 0.0208333 > V_n,$$

which is a contradiction, and so we can assume that $a_1 < \frac{1}{3}$. Next, assume that $a_n \leq \frac{2}{3}$. Then,

$$V_n \geq \int_{[\frac{2}{3}, 1]} (x - \frac{2}{3})^2 dP = \frac{1}{2} \int_{[\frac{2}{3}, 1]} (x - \frac{2}{3})^2 dP_2 = \frac{1}{45} = 0.0222222 > V_n,$$

which leads to a contradiction, and so we can assume that $\frac{2}{3} < a_n$. Thus, we see that $\alpha_n \cap [0, \frac{1}{3}] \neq \emptyset$ and $\alpha_n \cap (\frac{2}{3}, 1] \neq \emptyset$, which proves the proposition. \square

Proposition 6.8. *For $n \geq 2$, let α_n be an optimal set of n -means for P . Then, α_n does not contain any point from the open interval $(\frac{1}{3}, \frac{2}{3})$. Moreover, the Voronoi region of any point from $\alpha_n \cap J$ does not contain any point from K , and the Voronoi region of any point from $\alpha_n \cap K$ does not contain any point from J .*

Proof. By Proposition 6.7, the statement of the proposition is true for $n = 2$. Now, we prove it for $n = 3$. Consider the set of three points $\beta_3 := \{\frac{1}{6}, \frac{17}{24}, \frac{23}{24}\}$. Then,

$$\int \min_{a \in \beta_3} \|x - a\|^2 dP = \frac{1}{2} \left(\int_J (x - \frac{1}{6})^2 dP_1 + \int_{K_1} (x - \frac{17}{24})^2 dP_2 + \int_{K_2} (x - \frac{23}{24})^2 dP_2 \right) = \frac{43}{5760}.$$

Since V_3 is the quantization error for three-means, we have $V_3 \leq \frac{43}{5760} = 0.00746528$. Let $\alpha_3 := \{a_1, a_2, a_3\}$ be an optimal set of three-means such that $0 < a_1 < a_2 < a_3 < 1$. By Proposition 6.7, we have $a_1 < \frac{1}{3}$ and $\frac{2}{3} < a_3$. Suppose that $a_2 \in (\frac{1}{3}, \frac{2}{3})$. The following two cases can arise:

Case 1. $\frac{1}{3} < a_2 \leq \frac{1}{2}$.

Then, $\frac{1}{2}(a_2 + a_3) > \frac{2}{3}$ implying $a_3 > \frac{4}{3} - a_2 \geq \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$. Using an equation similar to (3), we can show that for $\frac{5}{6} < a_3 < 1$, the error $\frac{1}{2} \int_K (x - a_3)^2 dP_2$ is minimum if P -almost surely, $a_3 = \frac{5}{6}$, and the minimum value is $\frac{1}{120}$. Thus,

$$V_3 \geq \frac{1}{2} \int_K (x - \frac{5}{6})^2 dP_2 = \frac{1}{120} = 0.00833333 > V_3,$$

which is a contradiction.

Case 2. $\frac{1}{2} \leq a_2 < \frac{2}{3}$.

Then, $\frac{1}{2}(a_1 + a_2) < \frac{1}{3}$ implying $a_1 < \frac{2}{3} - a_2 \leq \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$. Similar in Case 1, for $0 < a_1 < \frac{1}{6}$, the error $\frac{1}{2} \int_J (x - a_1)^2 dP_1$ is minimum if P -almost surely, $a_1 = \frac{1}{6}$, and the minimum value is $\frac{1}{144}$. Thus,

$$V_3 \geq \frac{1}{144} + \frac{1}{2} \int_{K_1} (x - \frac{2}{3})^2 dP_2 = \frac{11}{1440} = 0.00763889 > V_3,$$

which leads to a contradiction.

Thus, by Case 1 and Case 2, we see that α_3 does not contain any point from $(\frac{1}{3}, \frac{2}{3})$. We now prove the proposition for all $n \geq 4$. Consider the set of four points $\beta_4 := \{\frac{1}{18}, \frac{5}{18}, \frac{17}{24}, \frac{23}{24}\}$. The distortion error due to the set β_4 is given by

$$\int \min_{a \in \beta_4} \|x - a\|^2 dP = \frac{1}{2}(V_2(P_1) + V_2(P_2)) = \frac{67}{51840} = 0.00129244.$$

Since V_n is the quantization error for n -means for all $n \geq 4$, we have $V_n \leq V_4 \leq 0.00129244$. Let $j = \max\{i : a_i < \frac{2}{3} \text{ for all } 1 \leq i \leq n\}$. Then, $a_j < \frac{2}{3}$. We need to show that $a_j < \frac{1}{3}$. For the sake of contradiction, assume that $a_j \in (\frac{1}{3}, \frac{2}{3})$. Then, two cases can arise:

Case A. $\frac{1}{3} < a_j \leq \frac{1}{2}$.

Then, $\frac{1}{2}(a_j + a_{j+1}) > \frac{2}{3}$ implying $a_{j+1} > \frac{4}{3} - a_j \geq \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$, and so,

$$V_n \geq \frac{1}{2} \int_{K_1} (x - \frac{5}{6})^2 dP_2 = \frac{1}{240} = 0.00416667 > V_n,$$

which leads to a contradiction.

Case B. $\frac{1}{2} \leq a_j \leq \frac{2}{3}$.

Then, $\frac{1}{2}(a_{j-1} + a_j) < \frac{1}{3}$ implying $a_{j-1} < \frac{2}{3} - a_j \leq \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$, and so,

$$V_n \geq \frac{1}{2} \int_{J_2} (x - \frac{1}{6})^2 dP_1 = \frac{1}{288} = 0.00347222 > V_n,$$

which gives a contradiction.

Thus, by Case A and Case B, we can assume that $a_j \leq \frac{1}{3}$. If the Voronoi region of any point from $\alpha_n \cap J$ contains points from K , then we must have $\frac{1}{2}(a_j + a_{j+1}) > \frac{2}{3}$ implying $a_{j+1} > \frac{4}{3} - a_j \geq \frac{4}{3} - \frac{1}{3} = 1$, which is a contradiction since $a_{j+1} < 1$. Similarly, the Voronoi region of any point from $\alpha_n \cap K$ does not contain any point from J . Thus, the proof of the proposition is complete. \square

Note 6.9. From Proposition 6.7 and Proposition 6.8, it follows that for $n \geq 2$, if an optimal set α_n contains n_1 elements from J and n_2 elements from K , then $n = n_1 + n_2$. In that case, we write $\alpha_n := \alpha_{(n_1, n_2)}$ and $V_n := V_{(n_1, n_2)}$. Thus, $\alpha_n = \alpha_{(n_1, n_2)} = \alpha_{n_1}(P_1) \cup \alpha_{n_2}(P_2)$, and $V_n = V_{(n_1, n_2)} = \frac{1}{2}(V_{n_1}(P_1) + V_{n_2}(P_2))$.

Lemma 6.10. *Let α be an optimal set of two-means for P . Then, $\alpha = \alpha_{(1,1)}$, and the corresponding quantization error is $V_2 = \frac{5}{432} = 0.0115741$.*

Proof. Let $\alpha = \{a_1, a_2\}$ be an optimal set of two-means such that $0 < a_1 < a_2 < 1$. By Proposition 6.7, we have $a_1 < \frac{1}{3}$ and $\frac{2}{3} < a_2$ yielding $a_1 = \frac{1}{6}$, $a_2 = \frac{5}{6}$, i.e., $\alpha = \alpha_1(P_1) \cup \alpha_1(P_2)$, and $V_2 = \frac{11}{720} = 0.0152778$. Thus, the proof of the lemma is complete. \square

Lemma 6.11. *Let α be an optimal set of three-means. Then, $\alpha = \alpha_{(1,2)}$, and the corresponding quantization error is $V_3 = \frac{43}{5760} = 0.00746528$.*

Proof. Let α be an optimal set of three-means. By Proposition 6.7 and Proposition 6.8, we can assume that either $\alpha = \alpha_2(P_1) \cup \alpha_1(P_2)$, or $\alpha = \alpha_1(P_1) \cup \alpha_2(P_2)$. Since

$$\int \min_{a \in \alpha_1(P_1) \cup \alpha_2(P_2)} (x - a)^2 dP < \int \min_{a \in \alpha_2(P_1) \cup \alpha_1(P_2)} (x - a)^2 dP,$$

the set $\alpha = \alpha_1(P_1) \cup \alpha_2(P_2)$ forms an optimal set of three-means, and the corresponding quantization error is

$$V_3 = \int \min_{a \in \alpha_1(P_1) \cup \alpha_2(P_2)} (x - a)^2 dP = \frac{1}{2}(V_1(P_1) + V_2(P_2)) = \frac{43}{5760} = 0.00746528,$$

which yields the lemma. \square

Lemma 6.12. *Let α be an optimal set of four-means. Then, $\alpha = \alpha_{(2,2)}$, and the corresponding quantization error is $V_4 = \frac{67}{51840} = 0.00129244$.*

Proof. Let α be an optimal set of four-means. By Proposition 6.7 and Proposition 6.8, we can assume that either $\alpha = \alpha_3(P_1) \cup \alpha_1(P_2)$, $\alpha = \alpha_2(P_1) \cup \alpha_2(P_2)$, or $\alpha = \alpha_1(P_1) \cup \alpha_3(P_2)$. Among all these possible choices, we see that $\alpha = \alpha_2(P_1) \cup \alpha_2(P_2)$ gives the minimum distortion error, and hence, $\alpha = \alpha_2(P_1) \cup \alpha_2(P_2)$ is an optimal set of four-means, and the corresponding quantization error is $V_4 = \frac{1}{2}(V_2(P_1) + V_2(P_2)) = \frac{67}{51840} = 0.00129244$, which is the lemma. \square

Remark 6.13. Proceeding in the similar way, as Lemma 6.12, it can be proved that the optimal sets of n -means for $n = 5, 6, 7$, etc. are, respectively, $\alpha_{(3,2)}$, $\alpha_{(2^2,2)}$, $\alpha_{(2^2,3)}$, etc.

We now prove the following lemma.

Lemma 6.14. *Let $\alpha_{(2^{6n-4}, 2^{5n-4})}$ be an optimal set of $2^{6n-4} + 2^{5n-4}$ -means for P for some positive integer n . For $1 \leq i \leq 5$ and $1 \leq j \leq 6$, let $\ell_i, k_j \in \mathbb{N}$ be such that $1 \leq \ell_i \leq 2^{5n-4+(i-1)}$ and $1 \leq k_j \leq 2^{6n-4+(j-1)}$. Then, (i) $\alpha_{(2^{6n-4}, 2^{5n-4}+\ell_1)}$ is an optimal set of $2^{6n-4} + 2^{5n-4} + \ell_1$ -means; (ii) $\alpha_{(2^{6n-4}+k_1, 2^{5n-3})}$ is an optimal set of $2^{6n-4} + 2^{5n-3} + k_1$ -means; (iii) $\alpha_{(2^{6n-3}, 2^{5n-3}+\ell_2)}$ is an optimal set of $2^{6n-3} + 2^{5n-3} + \ell_2$ -means; (iv) $\alpha_{(2^{6n-3}+k_2, 2^{5n-2})}$ is an optimal set of $2^{6n-3} + 2^{5n-2} + k_2$ -means; (v) $\alpha_{(2^{6n-2}, 2^{5n-2}+\ell_3)}$ is an optimal set of $2^{6n-2} + 2^{5n-2} + \ell_3$ -means; (vi) $\alpha_{(2^{6n-2}+k_3, 2^{5n-1})}$ is an optimal set of $2^{6n-2} + 2^{5n-1} + k_3$ -means; (vii) $\alpha_{(2^{6n-1}, 2^{5n-1}+\ell_4)}$ is an optimal set of $2^{6n-1} + 2^{5n-1} + \ell_4$ -means; (viii) $\alpha_{(2^{6n-1}+k_4, 2^{5n})}$ is an optimal set of $2^{6n-1} + 2^{5n} + k_4$ -means; (ix) $\alpha_{(2^{6n}, 2^{5n}+\ell_5)}$ is an optimal set of $2^{6n} + 2^{5n} + \ell_5$ -means; (x) $\alpha_{(2^{6n}+k_5, 2^{5n+1})}$ is an optimal set of $2^{6n} + 2^{5n+1} + k_5$ -means; and (xi) $\alpha_{(2^{6n+1}+k_6, 2^{5n+1})}$ is an optimal set of $2^{6n+1} + 2^{5n+1} + k_6$ -means.*

Proof. By Remark 6.13, it is known that $\alpha_{(2^{6n-4}, 2^{5n-4})}$ is an optimal set of $2^{6n-4} + 2^{5n-4}$ -means for $n = 1$. So, we can assume that $\alpha_{(2^{6n-4}, 2^{5n-4})}$ is an optimal set of $2^{6n-4} + 2^{5n-4}$ -means for P for some positive integer n . Recall that $\alpha_{(n_1, n_2)}$ is an optimal set of $n_1 + n_2$ -means, and contains n_1 elements from C and n_2 elements from D , and so, an optimal set of $n_1 + n_2 + 1$ -means must contain at least n_1 elements from C , and at least n_2 elements from D . For all $n \geq 1$, since

$$\frac{1}{2}(V_{2^{6n-4}}(P_1) + V_{2^{5n-4}+1}(P_2)) < \frac{1}{2}(V_{2^{6n-4}+1}(P_1) + V_{2^{5n-4}}(P_2)),$$

we can assume that $\alpha_{(2^{6n-4}, 2^{5n-4}+\ell_1)}$ is an optimal set of $2^{6n-4} + 2^{5n-4} + \ell_1$ -means for $\ell_1 = 1$. Having known $\alpha_{(2^{6n-4}, 2^{5n-4}+1)}$ as an optimal set of $2^{6n-4} + 2^{5n-4} + 1$ -means, we see that

$$\frac{1}{2}(V_{2^{6n-4}}(P_1) + V_{2^{5n-4}+2}(P_2)) < \frac{1}{2}(V_{2^{6n-4}+1}(P_1) + V_{2^{5n-4}+1}(P_2)),$$

and so, $\alpha_{(2^{6n-4}, 2^{5n-4}+\ell_1)}$ is an optimal set of $2^{6n-4} + 2^{5n-4} + \ell_1$ -means for $\ell_1 = 2$. Proceeding in this way, inductively, we can show that $\alpha_{(2^{6n-4}, 2^{5n-4}+\ell_1)}$ is an optimal set of $2^{6n-4} + 2^{5n-4} + \ell_1$ -means for $1 \leq \ell_1 \leq 2^{5n-4}$. Thus, (i) is true. Now, by (i), we see that $\alpha_{(2^{6n-4}, 2^{5n-3})}$ is an optimal set of $2^{6n-4} + 2^{5n-3}$ -means. Then, proceeding in the same way as (i) we can show that (ii) is true. Similarly, we can prove the statements from (iii) to (xi). Thus, the lemma is yielded. \square

Proposition 6.15. *The sets $\alpha_{(2^{6n-4}, 2^{5n-4})}$, $\alpha_{(2^{6n-4}, 2^{5n-3})}$, $\alpha_{(2^{6n-3}, 2^{5n-3})}$, $\alpha_{(2^{6n-3}, 2^{5n-2})}$, $\alpha_{(2^{6n-2}, 2^{5n-2})}$, $\alpha_{(2^{6n-2}, 2^{5n-1})}$, $\alpha_{(2^{6n-1}, 2^{5n-1})}$, $\alpha_{(2^{6n-1}, 2^{5n})}$, $\alpha_{(2^{6n}, 2^{5n})}$, $\alpha_{(2^{6n}, 2^{5n+1})}$, $\alpha_{(2^{6n+1}, 2^{5n+1})}$, and $\alpha_{(2^{6n+2}, 2^{5n+1})}$ are optimal sets for all $n \in \mathbb{N}$.*

Proof. By Remark 6.13, it is known that $\alpha_{(2^{6n-4}, 2^{5n-4})}$ is an optimal set of $2^{6n-4} + 2^{5n-4}$ -means for $n = 1$. Then, by Lemma 6.14, it follows that $\alpha_{(2^{6n-4}, 2^{5n-4})}$ is an optimal set of $2^{6n-4} + 2^{5n-4}$ -means for $n = 2$, and so, applying Lemma 6.14 again, we can say that $\alpha_{(2^{6n-4}, 2^{5n-4})}$ is an optimal set of $2^{6n-4} + 2^{5n-4}$ -means for $n = 3$. Thus, by induction, $\alpha_{(2^{6n-4}, 2^{5n-4})}$ are optimal sets of $2^{6n-4} + 2^{5n-4}$ -means for all $n \geq 2$. Hence, by Lemma 6.14, the statement of the proposition is true. \square

Remark 6.16. Because of Lemma 6.3, Lemma 6.10, Lemma 6.11, Lemma 6.12, and Remark 6.13, the optimal sets of n -means are known for all $1 \leq n \leq 6$. To determine the optimal sets of n -means for any $n \geq 6$, let $\ell(n)$ be the least positive integer such that $2^{6\ell(n)-4} + 2^{5\ell(n)-4} \leq n < 2^{6(\ell(n)+1)-4} + 2^{5(\ell(n)+1)-4}$. Then, using Lemma 6.14, we can determine n_1 and n_2 with $n = n_1 + n_2$ so that $\alpha_n = \alpha_{(n_1, n_2)}$ gives an optimal set of n -means. Once n_1 and n_2 are known, the corresponding quantization error is obtained by using the formula $V_n = \frac{1}{2}(V_{n_1}(P_1) + V_{n_2}(P_2))$.

6.17. Asymptotics for the n th quantization error $V_n(P)$. In this subsection, we investigate the quantization dimension and the quantization coefficients for the mixed distribution P . Let β_1 be the Hausdorff dimension of the Cantor set C generated by S_1 and S_2 , and β_2 be the Hausdorff dimension of the Cantor set D generated by T_1 and T_2 . Then, $\beta_1 = \frac{\log 2}{\log 3}$ and $\beta_2 = \frac{1}{2}$. If $D(P_i)$ are the quantization dimensions of P_i for $i = 1, 2$, then it is known that $D(P_1) = \beta_1$ and $D(P_2) = \beta_2$ (see [GL2]).

Theorem 6.17.1. *Let $D(P)$ be the quantization dimension of the mixed distribution $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$. Then, $D(P) = \max\{D(P_1), D(P_2)\}$.*

Proof. Define $F(n) := 2^{5n-4}(2^n + 1) = 2^{6n-4} + 2^{5n-4}$, where $n \in \mathbb{N}$. Notice that $F(n) \geq F(1) = 6$. For $n \in \mathbb{N}$, $n \geq 6$, let $\ell(n)$ be the least positive integer such that $F(\ell(n)) \leq n < F(\ell(n) + 1)$. Then, $V_{F(\ell(n)+1)} < V_n \leq V_{F(\ell(n))}$. Thus, we have

$$\frac{2 \log(F(\ell(n)))}{-\log(V_{F(\ell(n)+1)})} < \frac{2 \log n}{-\log V_n} < \frac{2 \log(F(\ell(n) + 1))}{-\log(V_{F(\ell(n))})}$$

Notice that

$$V_{F(n)} = V_{(2^{6n-4}, 2^{5n-4})} = \frac{1}{2} \left(V_{2^{6n-4}}(P_1) + V_{2^{5n-4}}(P_2) \right) = \frac{1}{240} (2^{17-20n} + 5 \cdot 3^{7-12n}).$$

Then,

$$\begin{aligned} \lim_{\ell(n) \rightarrow \infty} \frac{2 \log(F(\ell(n)))}{-\log(V_{F(\ell(n)+1)})} &= \lim_{\ell(n) \rightarrow \infty} \frac{2 \log(2^{6\ell(n)-4} + 2^{5\ell(n)-4})}{\log 240 - \log(2^{-3-20\ell(n)} + 5 \cdot 3^{-5-12\ell(n)})} \left(\frac{\infty}{\infty} \text{ form} \right) \\ &= \lim_{\ell(n) \rightarrow \infty} \frac{\frac{2^{6\ell(n)-4} 12 \log 2 + 2^{5\ell(n)-4} 10 \log 2}{2^{6\ell(n)-4} + 2^{5\ell(n)-4}}}{\frac{2^{-3-20\ell(n)} 20 \log 2 + 5 \cdot 3^{-5-12\ell(n)} 12 \log 3}{2^{-3-20\ell(n)} + 5 \cdot 3^{-5-12\ell(n)}}} = \frac{\log 2}{\log 3}, \end{aligned}$$

and similarly,

$$\lim_{\ell(n) \rightarrow \infty} \frac{2 \log(F(\ell(n) + 1))}{-\log(V_{F(\ell(n))})} = \frac{\log 2}{\log 3}.$$

Since $\ell(n) \rightarrow \infty$ whenever $n \rightarrow \infty$, we have $\frac{\log 2}{\log 3} \leq \liminf_n \frac{2 \log n}{-\log V_n} \leq \limsup_n \frac{2 \log n}{-\log V_n} \leq \frac{\log 2}{\log 3}$ implying the fact that the quantization dimension of the mixed distribution P exists and equals β_1 , i.e., $D(P) = D(P_1)$. Since $D(P_1) = \beta_1 > \beta_2 = D(P_2)$, we have $D(P) = \max\{D(P_1), D(P_2)\}$. Thus, the proof of the theorem is complete. \square

Theorem 6.17.1 verifies the following well-known proposition in [L] for $d = 1$ and $r = 1$.

Proposition 6.17.2. *(see [L, Theorem 2.1]) Let $0 < r < +\infty$, and let P_1 and P_2 be any two Borel probability measures on \mathbb{R}^d such that $D_r(P_1)$ and $D_r(P_2)$ both exist. If $P = pP_1 + (1-p)P_2$, where $0 < p < 1$, then $D_r(P) = \max\{D_r(P_1), D_r(P_2)\}$.*

Theorem 6.17.3. *Quantization coefficient for the mixed distribution $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$ does not exist.*

Proof. By Theorem 6.17.1, the quantization dimension of the mixed distribution exists and equals β_1 , where $\beta_1 = \frac{\log 2}{\log 3}$. To prove the theorem it is enough to show that the sequence $\left(n^{\frac{2}{\beta_1}} V_n(P)\right)_{n \geq 1}$ has at least two different accumulation points. By Lemma 6.14 (i), it is known that $\alpha_{(2^{6n-4}, 2^{5n-4})}$ is an optimal set of $2^{6n-4} + 2^{5n-4}$ -means. Again, by Lemma 6.14 (ii), it is known that $\alpha_{(2^{6n-4}+2^{6n-5}, 2^{5n-3})}$ is an optimal set of $2^{6n-4} + 2^{6n-5} + 2^{5n-3}$ -means. Write $F(n) := 2^{6n-4} + 2^{5n-4}$, and $G(n) := 2^{6n-4} + 2^{6n-5} + 2^{5n-3}$ for $n \in \mathbb{N}$. Recall that

$$V_{F(n)} = V_{(2^{6n-4}, 2^{5n-4})} = \frac{1}{2} \left(V_{2^{6n-4}}(P_1) + V_{2^{5n-4}}(P_2) \right) = \frac{1}{240} (2^{17-20n} + 5 \cdot 3^{7-12n}),$$

$$V_{G(n)} = V_{(2^{6n-4}+2^{6n-5}, 2^{5n-3})} = \frac{1}{2} \left(V_{2^{6n-4}+2^{6n-5}}(P_1) + V_{2^{5n-3}}(P_2) \right) = \frac{1}{15} 2^{9-20n} + \frac{5}{16} 81^{1-3n}.$$

Notice that $(2^{6n})^{\frac{2}{\beta_1}} = 2^{\frac{12n \log 3}{\log 2}} = 3^{12n}$ and $\lim_{n \rightarrow \infty} \left(\frac{3^{12}}{2^{20}}\right)^n = 0$, and so, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} F(n)^{\frac{2}{\beta_1}} V_{F(n)}(P) &= \lim_{n \rightarrow \infty} (2^{6n-4} + 2^{5n-4})^{\frac{2}{\beta_1}} \frac{1}{240} (2^{17-20n} + 5 \cdot 3^{7-12n}) \\ &= \lim_{n \rightarrow \infty} 3^{12n} \left(\frac{1}{2^4} + \frac{1}{2^4} \cdot \frac{1}{2^n} \right)^{\frac{2}{\beta_1}} \frac{1}{240} (2^{17-20n} + 5 \cdot 3^{7-12n}) = 2^{-\frac{8}{\beta_1}} \frac{5 \cdot 3^7}{240} = \frac{1}{144} = 0.00694444, \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} G(n)^{\frac{2}{\beta_1}} V_{G(n)}(P) &= \lim_{n \rightarrow \infty} (2^{6n-4} + 2^{6n-5} + 2^{5n-3})^{\frac{2}{\beta_1}} \left(\frac{1}{15} \cdot 2^{9-20n} + \frac{5}{16} \cdot 81^{1-3n} \right) \\ &= \lim_{n \rightarrow \infty} 3^{12n} \left(\frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^3} \frac{1}{2^n} \right)^{\frac{2}{\beta_1}} \left(\frac{1}{15} \cdot 2^{9-20n} + \frac{5}{16} \cdot 81 \cdot 3^{-12n} \right) = \frac{5}{16} \cdot 3^{\frac{2 \log(3)}{\log(2)} - 6} = 0.0139496. \end{aligned}$$

Since $(F(n)^{\frac{2}{\beta_1}} V_{F(n)}(P))_{n \geq 1}$ and $(G(n)^{\frac{2}{\beta_1}} V_{G(n)}(P))_{n \geq 2}$ are two subsequences of $(n^{\frac{2}{\beta_1}} V_n(P))_{n \in \mathbb{N}}$ having two different accumulation points, we can say that the sequence $(n^{\frac{2}{\beta_1}} V_n(P))_{n \in \mathbb{N}}$ does not converge, in other words, the β_1 -dimensional quantization coefficient for P does not exist. This completes the proof of the theorem. \square

We now conclude the paper with the following remark.

Remark 6.18. Optimal quantization for a general probability measure, singular or nonsingular, is still open, which yields the fact that the optimal quantization for a mixed distribution taking any two probability measures is not yet known.

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